Implementing Union-Find

A set is an unordered collection of things.
One way to implement a set is with a doubly-linked list.

- makeset(x)
  - create a linked list with a single node containing x
  - O(1)
- union(x,y)
  - append y's list to x's list
  - O(1) if you have tail pointers – set x's tail's next to point to y's head
- find(x)
  - how to identify a set? – could use the head node as the representative of the set
  - given a node, it is O(size of list) to find the head of its list – follow prev pointers backwards from node to the head

→ total: O(n ✕ makeset + m log n + m ✕ find + n ✕ union)
= O(n + m log n + nm + n) = O(nm)
- (this is much worse than graph traversal, can we do better?)

Implementing Union-Find

Can we do better? Find is the slow part…

- what's better than O(n)? → O(1)
  - if every node also had a pointer directly to the head, find(x) could be done in constant time!

New implementation: singly-linked list with tail pointer and each node also pointing directly to the head.

(can actually store head pointers instead of prev pointers since the only reason to back up was to find the head)

- makeset(x)
  - O(1)
- union(x,y)
  - O(1) to append...but O(size of y) to update all head pointers in y
- find(x)
  - O(1)

→ total: O(n ✕ makeset + m log n + m ✕ find + n ✕ union)
= O(n + m log n + m + n^2) = O(m log n + n^2)
- (somewhat better...)

Implementing Union-Find

union by rank list implementation

Can we do better? Union is the slow part – what if we updated as few head pointers as possible?

- union(x,y)
  - O(1) to append the smaller list to the larger list...but still O(size of smaller list) to update head pointers in the smaller list
    - can store list sizes so it is possible to find the smaller list in O(1)

→ total: O(n ✕ makeset + m log n + m ✕ find + n ✕ union)
- observation: in the worst case O(size of smaller list) is O(n), but we know something about the series of unions
  - each time we union and the head pointer for a node is updated, the node is going into a set at least twice as big as it come from
  - this can happen at most log n times if there's a total of n elements
- thus n unions with appending the smaller list is O(n log n) instead of O(n^2)

= O(n + m log n + m + n log n) = O((n+m) log n)
- an improvement for sparse graphs!

Implementing Union-Find

O((n+m) log n) for Kruskal's algorithm is pretty good – can we do better?

Observation.

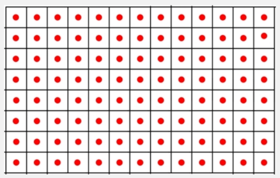
- sorting the edges by weight requires O(m log n), which will dominate O(n log n) as long as the graph is connected
  - improving the data structure will result in elapsed time gains, but not change the big-Oh

However…

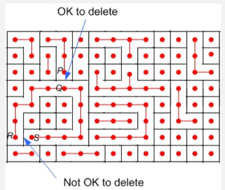
- union-find has applications beyond Kruskal's algorithm
  - greater efficiency in union-find operations may make a difference there

Maze Creation

In a good maze, every room is reachable from every other and there's only one possible path from start to goal.
How to generate a random maze?



start with all of the walls and every room in a separate set



while there is more than one set left
- choose a random wall
- if the rooms on either side belong to different sets, knock down the wall

Implementing Union-Find

We improved the implementation to speed up Kruskal's algorithm by making union the slow part instead of find.

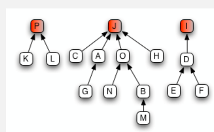
We've seen O(n)↔O(1) tradeoffs before...and sometimes could compromise on O(log n) for both.

Will that work here?

Observation.
- trees are often associated with O(log n) run times
  - the height can be as good as O(log n)

Implementing Union-Find

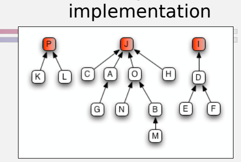
Idea:
Represent a set as a directed tree.



- makeset(x)
  - create a tree with a single node containing x
  - O(1)
- union(x,y)
  - find the roots of x's and y's trees
  - make y's root point to x's root
  - O(find)
- find(x)
  - use the root as the representative element
  - given a node, it is O(height of tree) to find the root of its tree

Implementing Union-Find

union by rank tree implementation



How tall are the trees?
- could be n – can we do better?
- union(x,y)
  - find the roots of x's and y's trees
  - make the root of the shorter tree point to the root of the taller tree
    - store rank(v) = height of subtree rooted at *v* for each node so the shorter tree can be found in O(1) time
    - only the rank of the taller tree's root may change as a result of the union – O(1) to update
- result is O(log n) height for each tree – so find is O(log n)
  - idea: height of merged tree only increases if the two trees are equally tall – that merge doubles the size of the tree so it can happen at most log n times

→ total: O(n × makeset + m log n + m × find + n × union)
= O(n + m log n + m log n + n log n) = O((n+m) log n)
- (no improvement over union-by-rank lists)

Implementing Union-Find

O(log n) find(x) isn't bad, but O(1) is still better...

Observation:

- could get O(1) find(x) if each node had a direct pointer to the root

But:

- updating these pointers during union is too expensive

Observation:

- find(x) locates the root for every node between x and the root

It seems a waste to throw that information away!

Implementing Union-Find

The best of both worlds: (*path compression*)

- union(x,y)
  - find the roots of x's and y's trees
  - make the root of the shorter tree point to the root of the taller tree
- find(x)
  - locate the root
  - update the pointers for every node on the path x→root to point directly to the root

Implementing Union-Find

Running time?

- find(x) and thus union(x,y) are still O(height of tree)

What's the height of the trees?

- path compression keeps the height of the trees short
- find(x) and union(x,y) are effectively O(1)

Implementing Union-Find

“Short”?! “Effectively”?!

- based on *amortized time*, not worst case
  - an individual operation may take longer, but if a sequence of *k* operations takes a total of O(k f(n)) time, we can say each operation is O(f(n)) amortized
- total time for *m* find(x) operations is O((m+n) log* n)
  - on average, O(n/m log* n) per find
  - with m > n (typical), this is O(log* n)
    - log* n = the number of successive log operations to bring n down to 1
    - extremely slow growing! (value < 5 for any value of *n* you might encounter, and thus is effectively constant time)

Union-Find Summary

- union-by-rank list implementation yields O((n+m) log n) for Kruskal's algorithm
  - O(1) makeset(x)
  - O(1) find(x)
  - O(n log n) for a series of *n* union(x,y)

- union-by-rank tree implementation with path compression yields O(m log n) for Kruskal's algorithm
  - O(1) makeset(x)
  - effectively O(1) find(x) and union(x,y)
    - the tree height is a very slow-growing log*
    - *amortized* over a series of operations

Both are an improvement over our initial O(nm) algorithm.