

You're planning to drive from Boston to Seattle on I-90 this summer, and have a GPS programmed with the locations of gas stations along the way. Assuming that your car can go 400 miles on a tank of gas, determine where you should stop for gas in order to make as few stops as possible.

Establish the problem.

- specifications – task, input, output, legal solution, optimal solution

task: identify the gas stations to stop at

input: locations of gas stations along I90

output: which stations to stop at

legal solution: no gaps between stations > 400 miles

optimal solution: fewest stops

- examples

Identify avenues of attack.

- known targets

generic $O(n)$ is good for loops

- approach

Series of choices.

- paradigms, patterns, and flavors

Paradigm: greedy.

Patterns: process input – for each gas station, do we stop?, produce output – repeatedly find the next stop

Flavors: **subset**, ordering, labeling

- greedy choices and counterexamples

info: location of gas station, can compute distance from another station

process input – do we stop? choice is based on distance to the next station

plausible way of using that info: stop if the next station is > 400 miles from the last stop

produce output – what is the next stop? choice is based on distance to future stations

plausible way of using that info: next stop is the farthest ahead within 400 miles

Define the algorithm.

- main steps

(produce output)

while not within 400 miles of Seattle

 the next stop is the farthest ahead within 400 miles

- exit condition

within 400 miles of Seattle

- setup

assume a full tank of gas in Boston (fill up the tank)

- wrapup

done with finding stops when we can get to seattle on the current tank – nothing to do after the loop

- special cases

what if there's a gap of more than 400 miles? – no solution

what if there's a tie for the farthest ahead within 400 miles? – pick any

if at any point the farthest ahead is 0 miles ahead (at our location) then no solution

- algorithm

Show termination and correctness.

- termination
 - measure of progress

distance remaining to seattle

- making progress

next stop is a station closer to seattle, so driving there decreases the distance remaining to seattle

the next stop is always closer because we go as far as we can, and that's only 0 if there's another station where we are and there's a gap beyond > 400 miles and that is no solution (handled special cases)

- the end is reached

each stop decreases the distance to seattle, so we'll eventually be within 400 miles

exit condition is within 400 miles of seattle

- correctness
 - loop invariant

produce output: after k output elements have been produced, we have a legal solution so far + staying ahead part

after k stops, no pair of successive stops are more than 400 miles apart and the alg is no farther from seattle than the optimal (after k stops)

- establish the loop invariant

$k=0$ (before first iteration)

alg and opt are both in boston – same distance to seattle, we're not behind

$k=1$ (after the first / before second)

alg picks farthest stop within 400 miles

opt picks a legal stop i.e. not more than 400 miles from boston

legal – we don't pick a stop more than 400 miles from our last location

staying ahead – we need: opt hasn't picked a stop closer to seattle than alg's

alg picked farthest legal choice, opt couldn't have picked a farther legal one – so we're staying ahead

- maintain the loop invariant

assume: after k stops, no pair of successive stops are more than 400 miles apart and the alg is no farther from seattle than the optimal (after k stops)

show: after $k+1$ stops, no pair of successive stops are more than 400 miles apart and the alg is no farther from seattle than the optimal (after $k+1$ stops)

strategy: by contradiction

assume: after $k+1$ stops, either there's a pair of successive stops more than 400 miles apart or the alg is farther from seattle than the optimal (after $k+1$ stops) [this is where things break]

legal: if we now have a gap, it's between stops k and $k+1$ but alg doesn't pick stops > 400 miles apart so not possible to now have a gap

staying ahead: if opt is now closer to seattle, $\text{dist}(\text{opt } k, \text{opt } k+1) > \text{dist}(\text{alg } k, \text{alg } k+1)$

opt always picks legal so $400 \geq \text{dist}(\text{opt } k, \text{opt } k+1) > \text{dist}(\text{alg } k, \text{alg } k+1)$

so alg could have pick opt's $k+1$ instead (which is farther)

- final answer

loop exits when we're within 400 miles of seattle (no more stops)

invariant gives us: after k stops, no pair of successive stops are more than 400 miles apart and the alg is no farther from seattle than the optimal (after k stops)

k = however many stops we've made before getting within 400 miles of seattle

how many stops does the opt need?

opt can't use fewer than k stops, or else it would have gotten to seattle earlier and we're always equal or ahead

opt can't need more than k stops, or else it's not opt because our k stop solution is legal

only choice – alg is opt because alg found same number of stops

Determine efficiency.

- implementation
- time and space
- room for improvement