1. (3 points) Write out each of the following sums in full, without using summation notation. You are not being asked to compute the value of the sums, just to expand each sum into a normal sum of several terms!

\[ \sum_{n=1}^{5} n \cdot 5^n \quad \sum_{i=1}^{7} (2i - 1) \quad \sum_{k=3}^{6} \frac{k}{k^2 + 1} \]

Answer:

a) \[ \sum_{n=1}^{5} n \cdot 5^n = 1 \cdot 5^1 + 2 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + 5 \cdot 5^5 \]

b) \[ \sum_{i=1}^{7} (2i - 1) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1) + (2 \cdot 6 - 1) + (2 \cdot 7 - 1) \]

c) \[ \sum_{k=3}^{6} \frac{k}{k^2 + 1} = \frac{3}{3^2 + 1} + \frac{4}{4^2 + 1} + \frac{5}{5^2 + 1} + \frac{6}{6^2 + 1} \]

2. (3 points) Use a proof by induction to show that for any integer \( n \geq 1 \), \( \sum_{i=1}^{n} (2i - 1) = n^2 \)

Answer:

**Base Case:** Let \( n = 1 \). Then \( \sum_{i=1}^{1} (2i - 1) = (2 \cdot 1 - 1) = 1 \), and \( n^2 = 1^2 = 1 \). So the statement is true for the base case.

**Inductive Case:** Let \( k \geq 1 \), and assume that the statement is true for \( n = k \). That is, assume that \( \sum_{i=1}^{k} (2i - 1) = k^2 \). We must show that the statement holds for \( n = k + 1 \), that is that \( \sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2 \). But

\[
\sum_{i=1}^{k+1} (2i - 1) = \left( \sum_{i=1}^{k} (2i - 1) \right) + (2 \cdot (k + 1) - 1) \\
= (k^2) + (2 \cdot (k + 1) - 1) \\
= k^2 + 2k + 2 - 1 \\
= k^2 + 2k + 1 \\
= (k + 1)^2
\]
3. (3 points) Use a proof by induction to show that for any integer \( n \geq 1 \), \( n^3 + 2n \) is a multiple of 3. (That is, there is some integer \( j \) such that \( n^3 + 2n = 3j \)).

Answer:

**Base Case:** Let \( n = 1 \). Then \( n^3 + 2n = 1^3 + 2 \cdot 1 = 3 = 3 \cdot 1 \). So \( n^3 + 2n \) is a multiple of 3 in this case.

**Inductive Case:** Let \( k \geq 1 \), and assume that \( n^3 + 2n \) is a multiple of 3 for \( n = k \). That is, there is an integer \( j \) such that \( k^3 + 2k = 3j \). We must show that \( n^3 + 2n \) is a multiple of 3 for \( n = k + 1 \). But for \( n = k + 1 \), we have

\[
(k + 1)^3 + 2(k + 1) = (k^3 + 3k^2 + 3k + 1) + (2k + 2) \\
= (k^3 + 2k) + (3k^2 + 3k + 1 + 2) \\
= (k^3 + 2k) + 3k^2 + 3k + 3 \\
= 3j + 3k^2 + 3k + 3 \\
= 3(j + k^2 + k + 1)
\]

Since \( j + k^2 + k + 1 \) is an integer, this shows that \((k + 1)^3 + 2(k + 1)\) is a multiple of 3.

4. (3 points) Use a proof by induction to show that the following method correctly finds the sum of array elements \( A[0]+ A[1]+ \cdots + A[N-1] \) for all \( N \geq 1 \).

```cpp
int recursive_sum( int[] A, int N ) {
    if ( N == 1 )
        return A[0];
    else {
        return A[N-1] + recursive_sum( A, N-1 );
    }
}
```

Answer:

**Base Case:** Let \( N = 1 \). In this case, the test in the if statement is true, so the function returns \( A[0] \). Since we are only taking the sum of one element, this is the correct answer.

**Inductive Case:** Suppose \( k \geq 1 \), and assume that the function is correct for \( N = k \). That is, \( \text{recursive}_\text{sum}(A,k) \) correctly computes the sum \( A[0]+A[1]+\cdots+A[k-1] \). We want to show that \( \text{recursive}_\text{sum}(A,k+1) \) correctly computes the sum \( A[0]+A[1]+\cdots+A[k] \).

Since \( k+1 > 1 \), when \( \text{recursive}_\text{sum}(A,k+1) \) is called, the test in the if statement is false, and the return value of the function is computed as \( A[(k+1)-1] + \text{recursive}_\text{sum}(A,(k+1)-1) \), which equals \( A[k] + \text{recursive}_\text{sum}(A,k) \). By the inductive hypothesis, \( \text{recursive}_\text{sum}(A,k) \) correctly returns the value of \( A[0]+A[1]+\cdots+A[k-1] \). Then \( A[k] \) is added to that return value, giving the correct answer, \( A[0]+A[1]+\cdots+A[k-1]+A[k] \).
5. (4 points) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; let $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$; and let $C = \{n \in \mathbb{Z} | -5 \leq n \leq 5\}$. Find the following sets. (For this exercise, you do not need to justify your answers.)

- **a)** $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18\}$
- **b)** $A \cap B = \{2, 4, 6, 8\}$
- **c)** $A \setminus B = \{1, 3, 5, 7, 9\}$
- **d)** $B \setminus A = \{10, 12, 14, 16, 18\}$
- **e)** $A \cap C = \{1, 2, 3, 4, 5\}$
- **f)** $\mathbb{N} \cup C = \{n \in \mathbb{Z} | n \geq -5\} = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- **g)** $\mathbb{N} \setminus C = \{n \in \mathbb{Z} | n > 5\} = \{6, 7, 8, 9, 10, 11, \ldots\}$
- **h)** $\mathbb{Z} \setminus A = \{n \in \mathbb{Z} | n \leq 0 \text{ or } n \geq 10\}$

(Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{Z}$ is the set of integers.)

Answer:

6. (2 points) Let $A$ be the set $A = \{\emptyset, a, \{a\}\}$. Write out the power set, $\mathcal{P}(A)$. (It has 8 elements. You do not have to justify your answer.)

Answer:

$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{a\}, \{\{a\}\}, \{a, \{a\}\}, \emptyset, \{a\}, \{\emptyset, a, \{a\}\}\}$

7. (3 points) Let $A$ be any set. What can you say about $A \cup A$? about $A \cap A$? about $A \setminus A$?

Justify your answer, either informally or by using the definitions of $\cup$, $\cap$, and $\setminus$.

Answer:

$A \cup A = A$, because $A \cup A = \{x | x \in A \lor x \in A\} = \{x | x \in A\} = A$. This uses the idempotent law of logic for $\lor$. Informally, adding all the elements of $A$ to all of the elements of $A$ just gives all of the elements of $A$.

$A \cap A = A$, because $A \cap A = \{x | x \in A \land x \in A\} = \{x | x \in A\} = A$. This uses the idempotent law of logic for $\land$. Informally, taking all the elements that are in both $A$ and $A$ gives all of the elements of $A$.

$A \setminus A = \emptyset$, because $A \setminus A = \{x | x \in A \land \neg(x \in A)\} = \emptyset$, because $x \in A \land \neg(x \in A)$ is of the logical form $p \land \neg p$, which is always false. Informally, removing from $A$ all of itself leaves nothing behind.

8. (3 points) Prove: If $A$, $B$, and $C$ are any sets and $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Answer:

Let $A$, $B$, and $C$ be arbitrary sets, and assume that $A \subseteq B$ and $A \subseteq C$. We want to show that $A \subseteq B \cap C$. Saying $A \subseteq B \cap C$ means by definition that for any $x$, if $x \in A$, then $x \in B \cap C$.

Let $x$ be an arbitrary element of $A$. We need to show that $x \in B \cap C$. Since $A \subseteq B$ and $x \in A$, we know that $x \in B$. Since $A \subseteq C$ and $x \in A$, we know that $x \in C$. Since $x \in B$ and $x \in C$, it follows by definition of intersection that $x \in B \cap C$. 
