This lab starts with one question on differential equations and ends with a question on using integration to analyze a signal into pure tones. Both questions represent real applications of the material that we have been learning. There are also a few problems about integration by parts.

All of the following problems should be turned in at next week's lab.

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1. Suppose that an object is dropped, with an initial velocity of zero, near the surface of the Earth. Let y be the number of feet it has fallen, as a function of time, t, measured in seconds. In a vacuum, where there is no air resistance, the object has an acceleration of 32 (feet per second per second), so that y satisfies the differential equation $y'' = 32$. (The acceleration is positive since we are measuring distance traveled downwards.) This differential equation has the general solution $y'' = 16t^2 + C_1 t + C_2$, where C_1 and C_2 are constants. In this case, since $y'(0) = 0$ and $y(0) = 0$, both constants are equal to zero, and so the distance fallen in t seconds is given by $y = 16t^2$. But this only applies when there is no air resistance. Air resistance adds an additional force that is proportional to the velocity of the object (or at least approximately so for most objects). The faster the object moves, the more air resistance. With air resistance taken into account, the distance fallen by the object satisfies the differential equation $y'' = 32 - ky'$, where k is a positive constant. You don't have the tools needed to solve this equation, so I will tell you the solution.

a) Why is there a minus sign rather than a plus sign in the equation $y'' = 32 - ky'$?

b) Show that $y(t) = \frac{32}{k}t + \frac{32}{k^2}(e^{-kt} - 1)$ is a solution of the differential equation $y'' = 32 - ky'$ with initial conditions $y(0) = y'(0) = 0$. Find explicit formulas for $y'(t)$ and $y''(t)$.

c) Describe in words what happens in the long run to the acceleration, velocity, and distance fallen, based on the formulas for these quantities. In particular, find the limit of the velocity as $t \to \infty$; this limit is called the *terminal velocity* of the object. Explain why this behavior makes sense physically.

d) Find a formula for the difference between the velocity of the object, $y'(t)$ and the terminal velocity of the object. Check that the formula is an example of exponential decay.

e) The value of the constant k depends on the particular object under discussion. Based on the information in this problem, explain why an ant dropped from a great height will be uninjured, while a fall from the same height would kill a person.

2. Each of the following three problems requires *both* integration by parts and a substitution. Compute each integral. Start with a substitution. (To avoid confusion, use z or θ, not u, as the variable in the substitution.)

a) $\displaystyle\int \sin(x)\cos(x)e^{\sin(x)}\,dx$ **b)** $\displaystyle\int \frac{\ln(\ln(x))}{x}\,dx$ **c)** $\displaystyle\int \sin(\sqrt{x}\,)\,dx$

3. Use a double integration by parts to evaluate the integral $\displaystyle\int \sin(\ln(x))\,dx$

4. It is possible to use the following identity to compute certain integrals:

$$\sin(mx)\sin(nx) = \frac{1}{2}\big(\cos((m-n)x) - \cos((m+n)x)\big)$$

Note that if $m = n$, this becomes:

$$\sin^2(nx) = \frac{1}{2}\big(1 - \cos(2nx)\big)$$

a) Use the first identity to compute the indefinite integral $\int \sin(12x)\sin(7x)\,dx$

b) Use these identities to show that $\frac{2}{\pi}\int_0^\pi \sin(mx)\sin(nx)\,dx$ is 0 when $n \neq m$ and that it is 1 when $n = m$, where n and m are positive integers.

c) Suppose that $f(x)$ is a function that has the form $f(x) = \displaystyle\sum_{n=1}^{K} a_n \sin(nx)$ for some positive integer K and some constants $a_1, a_2, \ldots, a_K$. Suppose that we know that $f(x)$ can be written in this form, but we don't know the constants a_n. Use the properties of integrals and the results from part **a)** to show that a_n can be computed as $a_m = \frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx$. This shows that, starting from the function without knowing the values of the constants a_n, we can use integration to discover their values.

(The function $\sin(nx)$ represents a "pure tone" or a "pure frequency" of n. $f(x)$ is a combination of these pure tones. This exercise shows how $f(x)$ can be analyzed to determine how much of each pure frequency it contains. This applies, for example, to a sound that is made up of a finite number of pure tones. This idea is the beginning of *digital signal analysis*, which is used in computer processing of sound waves and other signals. Extended to infinite sums, it leads to *Fourier analysis*, one of the core fields of applied mathematics.)