The final exam for this course is scheduled for Tuesday, May 9, at 1:30 PM, in our regular classroom. The exam includes everything that we have done in the course, with some emphasis on the material covered since the third test. Although the exam period is three hours long, most people should finish in less than two hours. The exam will be six or seven pages long, with the usual mix of problems, definitions, and essays. There will be at least one longer essay summarizing some central ideas in the Calculus.

As usual, you will not need a calculator for the test. A basic non-graphing calculator will be provided to you, and you will be permitted to use only the calculator that is provided. Scrap paper will also be provided. All you need is a pencil or pen.

Here are some terms and ideas covered after the third test:

- additional tests for convergence of series:
  - The Comparison Test
  - The Ratio Test
  - The Limit Comparison Test
  - The Root Test
  - The Alternating Series Test

- absolute convergence of series
- conditional convergence of series
- if a series converges absolutely, then it converges
- alternating series
- the remainder in a convergent alternating series
- power series
- radius of convergence of a power series
- interval of convergence of a power series
- using the ratio (or root) test to find the radius of convergence
- integration and differentiation of power series

\[ f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \] on an interval around \( a \), then \( c_n = \frac{f^{(n)}(a)}{n!} \)

Taylor series centered at \( a \) for a function \( f(x) \)
- the Taylor series for a function does not converge to that function in all cases
- Maclaurin series (just a Taylor series at 0)
- Taylor polynomial of order \( n \), centered at \( a \), for a function \( f(x) \)
Here are some major terms and ideas from earlier in the semester:

- Riemann sum; left, right, and midpoint Riemann sums
- Area under a curve
- Definite integral, defined as a limit of Riemann sums
- Properties of definite integrals
- How definite integrals relate to area
- The Fundamental Theorem of Calculus, part I and part II
- Method of substitution ("change of variables") for finding an indefinite integral
- Net change of a quantity $Q(t)$, given by $\int_a^b Q'(t) \, dt$
- Application to velocity, acceleration, displacement, distance traveled
- Area between curves
- Volume; volumes by slicing: integrating the cross-sectional area
- Volumes of revolution; the disk method, the washer method, and the shell method
- Integrating with respect to $y$ instead of with respect to $x$
- Length of a curve
- Change of variables in a definite integral
- Integration by parts
- Partial fractions
- Differential equations and initial value problems
- Solving separable first-order differential equations
- Improper integrals; convergence and divergence of improper integrals
- Infinite sequences and infinite series
- Partial sums of an infinite series
- Convergence and divergence of sequences and series
- Properties of sequences and series
- Geometric sequences and geometric series
- The Divergence Test
- The Integral Test
- $p$-series

Note: See also the study guides for the three tests, which you can find online.
Tests for convergence of infinite series...

**Geometric Series Test:** A geometric series with ratio \( r \) diverges if \( |r| \geq 1 \) and converges if \( |r| < 1 \). A convergent geometric series converges to \( \frac{a}{1-r} \) where \( a \) is the initial term and \( r \) is the ratio.

**Divergence Test** (also called the \( n^{th} \) term test): If \( \lim_{n \to \infty} a_n \neq 0 \) (including the case \( \lim_{n \to \infty} a_n \) does not exist) then the infinite series \( \sum_{n=0}^{\infty} a_n \) diverges.

**Integral Test:** Suppose that \( f(x) \) is a positive, decreasing function for \( x \geq 1 \), and suppose \( a_n = f(n) \) for \( n = 1, 2, 3, \ldots \). If the improper integral \( \int_1^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} a_n \) also converges. If the improper integral \( \int_1^{\infty} f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} a_n \) also diverges.

**p-series:** \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

**Alternating Series Test:** Suppose that \( \{a_n\}_{n=0}^{\infty} \) is a decreasing sequence of positive terms such that \( \lim_{n \to \infty} a_n = 0 \). Then \( \sum_{n=0}^{\infty} a_n \) converges.

**Comparison Test:** Suppose that \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) are infinite series of positive terms and that \( a_n \leq b_n \) for all \( n \). If \( \sum_{n=0}^{\infty} b_n \) converges, then \( \sum_{n=0}^{\infty} a_n \) also converges. If \( \sum_{n=0}^{\infty} a_n \) diverges, then \( \sum_{n=0}^{\infty} b_n \) also diverges.

**Limit Comparison Test:** Suppose that \( \sum_{n=0}^{\infty} a_n \) and \( \sum_{n=0}^{\infty} b_n \) are infinite series of positive terms and that \( 0 < \lim_{n \to \infty} \frac{a_n}{b_n} < \infty \). Then \( \sum_{n=0}^{\infty} a_n \) converges if and only if \( \sum_{n=0}^{\infty} b_n \) converges.

**Ratio Test:** Suppose that \( \sum_{n=0}^{\infty} a_n \) is an infinite series of positive terms. Let \( L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).

Assume the limit exists or is infinite. If \( L < 1 \), then the series converges. If \( L > 1 \) (including \( L = \infty \)), then the series diverges. (No information if \( L = 1 \).)

**Ratio Test:** Suppose that \( \sum_{n=0}^{\infty} a_n \) is an infinite series of positive terms, and let \( L = \lim_{n \to \infty} (a_n)^{1/n} \).

Assume the limit exists or is infinite. If \( L < 1 \), then the series converges. If \( L > 1 \) (including \( L = \infty \)), then the series diverges. (No information if \( L = 1 \).)

Note that the last four tests can be used to test for absolute convergence of any series, by taking the absolute value of the terms from the series.
Most of the final exam from Math 131 in Spring 2011:

1. For each of the following infinite series, determine whether the series converges or diverges. State clearly which rule you are applying and why it applies.

   a) \( \sum_{k=1}^{\infty} \frac{k^3}{k^6 + 4} \)
   
   b) \( \sum_{k=1}^{\infty} \frac{k!}{7^k \sqrt{k}} \)
   
   c) \( \sum_{k=1}^{\infty} \left( -\frac{3}{5} \right)^{k+1} \)
   
   d) \( \sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^2 + 1} \)

2. Suppose that \( g(x) \) is a function and that \( g(x) = \sum_{k=1}^{\infty} \frac{x^k}{k \cdot 5^k} \). Find a series that converges to \( g'(x) \), and then find \( g'(3) \).

3. Let \( f(x) = \sqrt{x} \). Find the Taylor polynomial of degree 3, centered at \( a = 1 \), for \( f(x) \).

4. Suppose that \( \{a_k\}_{k=1}^{\infty} \) is a sequence of positive numbers \( (a_k > 0) \), and that the series \( \sum_{k=0}^{\infty} a_k \) converges. Show that \( \sum_{k=0}^{\infty} a_k^2 \) also converges. (Hint: \( \left( \frac{1}{2} \right)^2 = \frac{1}{4} \).)

5. Find each of the following indefinite integrals.

   a) \( \int x \sin(2x^2 + 1) \, dx \)
   
   b) \( \int x \sin(2x + 1) \, dx \)
   
   c) \( \int \frac{2x \cos(x^2)}{\sin(x^2) + 2} \, dx \)
   
   d) \( \int \frac{5}{(x - 3)(2x - 1)} \, dx \)

6. Consider the region in the first quadrant bounded by the \( y \)-axis and by the curves \( y = 4 + x^2 \) and \( y = 2x^2 \).

   a) Write a Left Riemann Sum, using 4 subintervals, to approximate the area of the region.

   b) Use an integral to find the exact area of the region.

   c) Write but do not evaluate an integral that gives the volume of the solid that is generated when the region is rotated about the \( y \)-axis.

   d) Write but do not evaluate an integral that gives the volume of the solid that is generated when the region is rotated about the \( x \)-axis.

   e) Write but do not evaluate an integral that gives the volume of the solid that is generated when the region is rotated about the vertical line \( y = -2 \).

7. A point moves along a line. It starts with velocity zero at time \( t = 0 \). It accelerates with an acceleration given by \( a(t) = e^{-t} \) feet per second per second. How far has it moved at time \( t = 2 \) seconds?

8. The Fundamental Theorem of Calculus shows the relationship between antiderivatives and definite integrals. This theorem has two parts. Choose either part, state it, and explain its importance.

9. Throughout two terms of calculus, you have used limits in various ways. In each case, limits bring clarity and rigor to a problem that would be hard to deal with in any other way. Write an essay that describes some of the applications of limits in calculus. In each case, discuss the problem that is solved by using limits, and explain why limits are so essential to the solution.