Section 8.3

10. \( \lim_{n \to \infty} \frac{n^{12}}{3n^{13} + 4} = \frac{1}{3} \), since for the rational function \( \frac{x^{12}}{3x^{13} + 4} \),

\[ \lim_{x \to \infty} \frac{x^{12}}{3x^{13} + 4} = \lim_{x \to \infty} \frac{x^{12}}{3x^{13}} = \lim_{x \to \infty} \frac{1}{3} = \frac{1}{3} \]

("highest power rule")

or

\[ \lim_{x \to \infty} \frac{x^{12}}{3x^{13} + 4} = \lim_{x \to \infty} \frac{12x^{12}}{3 \cdot 12x^{12}} = \lim_{x \to \infty} \frac{1}{3} = \frac{1}{3} \]

(\( L'Hôpital's \) rule)

12. \( \lim_{n \to \infty} \frac{2e^n + 1}{e^n} = 2 \), since by \( L'Hôpital's \) rule,

\[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x} = \lim_{x \to \infty} \frac{2e^x}{e^x} = \lim_{x \to \infty} 2 = 2 \]

34. \( \lim_{n \to \infty} \frac{(-1)^n + 2}{2n^3 + n} = 0 \), since \( \lim_{n \to \infty} \frac{n^3}{2n^3 + n} = 0 \) \[ \text{[Problem 4, on Lab 12]} \]

\[ \text{or} \]

\[ \lim_{x \to \infty} \frac{x^2}{2x^3 + x} = \lim_{x \to \infty} \frac{1}{2x} = 0 \] (by the "highest power rule")

which means that both the positive and the negative terms in the series approach \( 0 \). Since all terms approach \( 0 \), the limit of the sequence is \( 0 \)

50. \( \lim_{n \to \infty} 2^{n+3} - 3^n = \lim_{n \to \infty} \frac{2^n \cdot 3^n}{3^n} = \lim_{n \to \infty} 2^n \left( \frac{2}{3} \right)^n = 0 \)

Since this is a geometric sequence with \( r = \frac{2}{3} \), and a geometric sequence converges to 0 if \( |r| < 1 \).
28. \[ \sum_{k=3}^{\infty} \frac{3 \cdot 4^k}{7^k} = \sum_{k=3}^{\infty} 3 \left( \frac{4}{7} \right)^k \] This is a geometric series with \( r = \frac{4}{7} \) and \( a = \frac{3 \cdot 4^3}{7^3} \), so it converges to
\[ \frac{a}{1-r} = \frac{3 \cdot 4^3}{7^3} \cdot \frac{7^3}{1 - \frac{4}{7}} = \frac{3 \cdot 4^3}{7^3} \cdot \frac{7^3}{3} = \frac{4^3}{7^2} = \frac{64}{49} \]

40. \[ \sum_{k=1}^{\infty} 3 \left( -\frac{1}{8^k} \right) = \sum_{k=1}^{\infty} 3 \left( -\frac{1}{8^k} \right) \] a geometric series with \( r = -\frac{1}{8} \) and \( a = -\frac{3}{8} \), which converges to
\[ \frac{a}{1-r} = \frac{-\frac{3}{8}}{1 - \left( -\frac{1}{8} \right)} = \frac{-\frac{3}{8}}{1 + \frac{1}{8}} = \frac{-3}{8^2 + 1} = \frac{-3}{513} \approx -\frac{1}{171} \]

46. \[ 0.27 = \frac{27}{100} + \frac{27}{(100)^2} + \frac{27}{(100)^3} + \frac{27}{(100)^4} + \ldots \] (Geometric series, \( r = \frac{1}{100}, a = \frac{27}{100} \))

\[ = \frac{\frac{27}{100}}{1 - \frac{1}{100}} = \frac{27}{100 - 1} = \frac{27}{99} = \frac{3}{11} \]

[Could also be done directly: Let \( x = 0.27272727\ldots \) Then
\[ 100x = 27.272727\ldots \] So, \( 100x - x = 27 \), \( 99x = 27 \)
and \( x = \frac{27}{99} = \frac{3}{11} \).]

56. \[ \sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right) = \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \ldots \]

The \( n \)-th partial sum is
\[ S_n = \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \ldots + \left( \frac{1}{n+2} - \frac{1}{n+3} \right) \]
\[ = \frac{1}{3} - \frac{1}{n+3} \]

So, \( \sum_{k=1}^{\infty} \left( \frac{1}{k+2} - \frac{1}{k+3} \right) = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( \frac{1}{3} - \frac{1}{n+3} \right) = \frac{1}{3} - 0 = \frac{1}{3} \)