1. Suppose that a power series $\sum_{n=0}^{\infty} c_n x^n$ converges on an open interval that contains zero. Then the power series defines a function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ on that interval. We can write this function out more explicitly as

$$f(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

So, a power series is something like an infinite polynomial. Recall also that you can take the derivative of a convergent power series term-by-term, just like a polynomial.

a) For the function $f(x)$ defined by the above power series, explain why $f(0) = c_o$.

b) Find a formula for $f'(x)$, written out like an infinite polynomial. Then find $f'(0)$.

c) Similarly, find $f''(0)$, and deduce that $c_2 = \frac{1}{2} f''(0)$.

d) Similarly, find $f'''(0)$, and deduce that $c_3 = \frac{1}{3 \cdot 2} f'''(0 = \frac{1}{3!} f'''(0)$.

e) Recall that $f^{(n)}$ is a notation for the n-th derivative of f. Find $f^{(4)}(0)$, and deduce that $c_4 = \frac{1}{4!} f^(4)(0)$.

f) Explain why $c_n = \frac{1}{n!} f^{(n)}(0)$ for all n.

g) Suppose that $g(x)$ is defined by $g(x) = \sum_{n=0}^{\infty} \frac{n x^n}{n^2+1}$. This series converges for $|x| < 1$. Find $g^{(10)}(0)$.

2. We have shown that the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x. Let $f(x)$ be the function defined by this power series. Show that $f'(x) = f(x)$ and that $f(0) = 1$. (It might be useful to write out the series as an infinite polynomial.) This means that f is a solution of the initial value problem $\frac{dy}{dx} = y$, $y(0) = 1$. We know that there is only one solution. What is it? What is f?

3. Recall that a series $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges. It *converges conditionally* if it converges but $\sum_{k=1}^{\infty} |a_k|$ does not converge. The other possibility is that the series $\sum_{k=1}^{\infty} a_k$ diverges—in that case, $\sum_{k=1}^{\infty} |a_k|$ automatically diverges as well.

Determine whether each of the following infinite series converges absolutely, or converges conditionally or diverges.

a) $\sum_{k=1}^{\infty} \frac{(-5)^n}{\sqrt{k}}$ **b)** $\sum_{k=2}^{\infty} \frac{k}{k^3 - 1}$ **c)** $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{5^k \sqrt{k}}$ **d)** $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$

e) $\sum_{k=1}^{\infty} \frac{2}{3^k + k^3}$ **f)** $\sum_{k=1}^{\infty} \frac{10000^k}{k^k}$ **g)** $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ **h)** $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k^5}$

(Hint for part **d)**: Show that $\ln(k) < \sqrt{k}$ for all sufficiently large k.)