Answer 1. a) \{0, 4, 16, 36, 64, 100, \ldots\} = \{(x^2 : x \in \mathbb{Z} \text{ and } x \text{ is even}\}. The elements are the squares of the even integers: 0 = 0^2, 4 = 2^2 = (-2)^2, 16 = 4^2 = (-4)^2, etc. (We can use all of the integers here because the square of a negative integer is positive. This could also be written, for example, as \{(2x)^2 : x \in \mathbb{Z} \text{ and } x \geq 0\}; here, by using \(2x\), we only get the squares of even numbers.)

b) \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\} = \{2 + 5n : n \in \mathbb{Z}\}. The numbers in the set are separated by 5, so we can get all the elements in the set by starting with 2 and adding positive or negative multiples of 5. In fact, we could start with any element; for example, the set can be written \{-8 + 5n : n \in \mathbb{Z}\}.

c) \{\ldots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27 \ldots\} = \{3^n : n \in \mathbb{Z}\}. The elements in the set are powers of three, 1 = 3^0, 3 = 3^1, 9 = 3^2, 27 = 3^3, and so on, and \(\frac{1}{3} = 3^{-1}, \frac{1}{9} = 3^{-2}\), and so on.

Answer 2. \(\{n \in \mathbb{Z} : 2 < n < 5\} \times \{n \in \mathbb{Z} : |n| = 5\} = \{3, 4\} \times \{-5, 5\} = \{(3, 5), (3, -5), (4, 5), (4, -5)\}

Answer 3. a) \(|\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(B)| = 2^n \cdot 2^m = 2^{n+m}\). This uses the fact that \(|X \times Y| = |X| \cdot |Y|\) and the fact that \(|\mathcal{P}(X)| = 2^{|X|}\).

b) \(|\{X : X \in \mathcal{P}(A) \text{ and } |X| \leq 1\}| = n + 1\). The set \(\mathcal{P}(A)\) contains all subsets of \(A\). The set \(X\) consists of the subsets of \(A\) that have zero elements or one element. The only subset with zero elements is the empty set, so there is one subset with cardinality 0. For each of the \(n\) elements of \(A\), we get a subset that contains just that one element, which gives a total of \(n\) subsets with cardinality one. This gives \(n + 1\) subsets with cardinality zero or 1.

Answer 4. Yes, it is always true that if \(A \subseteq B\), then \(\mathcal{P}(A) \subseteq \mathcal{P}(B)\). If \(X\) is any element of \(\mathcal{P}(A)\), then \(X\) is a subset of \(A\). But anything in \(A\) is also in \(B\), so that means that \(X\) is also a subset of \(B\). But saying \(X\) is a subset of \(B\) means that \(X\) an element of \(\mathcal{P}(B)\).

Answer 5. \(\overline{A} = A\). Saying \(x \in \overline{A}\) means that \(x\) is not in the complement of \(A\); that is, \(x\) is not outside of \(A\). But that is the same as saying that \(x\) is inside \(A\).

Answer 6. \(\bigcup_{i \in \mathbb{N}} A_i = \{n \in \mathbb{Z} : n \text{ is even}\}\), since every even number is in one of the sets \(A_n\). (Any even integer \(2k\) is in \(A_{|k|}\).) And \(\bigcap_{i \in \mathbb{N}} A_i = \{0\}\), since 0 is the only number that is in \(A_n\) for all \(n\). (In fact, \(A_1 \cap A_2 = \{2, 0, 2\} \cap \{4, 0, 4\}\), so the intersection of the first two sets is already just \(\{0\}\).)
**Answer 7.** \( \bigcup_{i \in \mathbb{N}} [0, i+1] = [0, \infty) \), since every non-negative real number is in one of the sets, and the sets contain only non-negative real numbers. And \( \bigcap_{i \in \mathbb{N}} [0, i+1] = [0, 2] \), since all the sets contain the interval \([0, 2]\) and the intersection can’t be bigger than \([0, 2]\) because \([0, 2]\) is one of the sets that is being intersected.

**Answer 8.** If \( \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} A_\alpha \), then all of the sets \( A_\alpha \) must be equal, and each of them is equal to the intersection. Let \( B \) be the intersection, which is the same as the union. Let \( A_\alpha \) be one of the sets. When you take the union of some sets, every one of those sets is contained in the union, so \( A_\alpha \subseteq B \). When you take the intersection of some sets, every one of those sets contains the intersection, so \( B \subseteq A_\alpha \). Saying \( A_\alpha \subseteq B \) and \( B \subseteq A_\alpha \) is the same as saying \( A_\alpha = B \). That is, every one of the sets \( A_\alpha \) is equal to \( B \).

**Answer 9.** Yes. If \( J \neq \emptyset \) and \( J \subseteq I \), then \( \bigcap_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in J} A_\alpha \). If \( x \) is in the first intersection, then \( x \in A_\alpha \) for every \( \alpha \in I \). But since \( J \subset I \), it follows that \( x \in A_\alpha \) for every \( \alpha \in J \), and that means that \( x \) is in the second intersection. (We only need \( J \neq \emptyset \) because the intersection of zero sets has not been defined.)

**Answer 10.** If \( C = \emptyset \), then \( X \times C = \emptyset \) for any set \( X \). So, for example \( \{1\} \times \emptyset = \{2\} \times \emptyset \) even though \( \{1\} \neq \{2\} \). However, if \( A \times C = B \times C \) and \( C \neq \emptyset \), then \( A \) must equal \( B \). To see this, note that if \( C \) is not empty, then there exists some element \( c \in C \). Then for any \( a \in A \), we have \((a, c) \in A \times C \). But \( A \times C = B \times C \), so we also have \((a, c) \in B \), which means that \( a \) must be in \( B \) also. (Another way of saying this: If \( C \neq \emptyset \), then \( A \) is clearly equal to the set of first coordinates of ordered pairs in \( A \times C \), and \( B \) is equal to the set of first coordinates of ordered pairs in \( B \times C \). So \( A \times C = B \times C \) will imply \( A = B \).)