Math 135, Fall 2019, Homework 1 Answers

Answer 1. a) $\{0, 4, 16, 36, 64, 100, \dots\} = \{(x^2 : x \in \mathbb{Z} \text{ and } x \text{ is even}\}$. The elements are the squares of the even integes: $0 = 0^2$, $4 = 2^2 = (-2)^2$, $16 = 4^2 = (-4)^2$, etc. (We can use all of the integers here because the square of a negative integer is positive. This could also be written, for example, as $\{(2x)^2 : x \in \mathbb{Z} \text{ and } x \geq 0\}$; here, by using $(2x)^2$, we only get the squares of even numbers.)

b) $\{\dots, -8, -3, 2, 7, 12, 17, \dots\} = \{2 + 5n : n \in \mathbb{Z}\}$. The numbers in the set are separated by 5, so we can get all the elements in the set by starting with 2 and adding positive or negative multiples of 5. In fact, we could start with any element; for example, the set can be written $\{-8 + 5n : n \in \mathbb{Z}\}$.

c) $\{\dots, \frac{1}{27}, \frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27 \dots\} = \{3^n : n \in \mathbb{Z}\}$. The elements in the set are powers of three, $1 = 3^0$, $3 = 3^1$, $9 = 3^2$, $27 = 3^3$, and so on, and $\frac{1}{3} = 3^{-1}$, $\frac{1}{9} = 3^{-2}$, and so on.

Answer 2. $\{n \in \mathbb{Z} : 2 < n < 5\} \times \{n \in \mathbb{Z} : |n| = 5\} = \{3, 4\} \times \{-5, 5\} = \{(3, 5), (3, -5), (4, 5), (4, -5)\}$

Answer 3. a) $|\mathscr{P}(A) \times \mathscr{P}(B)| = |\mathscr{P}(A)| \cdot |\mathscr{P}(B)| = 2^n \cdot 2^m = 2^{n+m}$. This uses the fact that $|X \times Y| = |X| \cdot |Y|$ and the fact that $|\mathscr{P}(X)| = 2^{|X|}$.

b) $|\{X : X \in \mathscr{P}(A) \text{ and } |X| \leq 1\}| = n + 1$. The set $\mathscr{P}(A)$ contains all subsets of A. The set X consists of the subsets of A that have zero elements or one element. The only subset with zero elements is the empty set, so there is one subset with cardinality 0. For each of the n elements of A, we get a subset that contains just that one element, which gives a total of n subsets with cardinality one. This gives $n + 1$ subsets with cardinality zero or 1.

Answer 4. Yes, it is always true that if $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. If X is any element of $\mathscr{P}(A)$, then X is a subset of A. But anything in A is also in B, so that means that X is also a subset of B. But saying X is a subset of B means that X an element of $\mathscr{P}(B)$.

Answer 5. $\overline{\overline{A}} = A$. Saying $x \in \overline{\overline{A}}$ means that x is **not** in the complement of A; that is, x is **not** outside of A. But that is the same as saying that x **is** inside A.

Answer 6. $\bigcup_{i \in \mathbb{N}} A_i = \{n \in \mathbb{Z} : n \text{ is even}\}$, since every even number is in one of the sets A_n. (Any even integer $2k$ is in $A_{|k|}$.) And $\bigcap_{i \in \mathbb{N}} A_i = \{0\}$, since 0 is the only number that is in A_n for all n. (In fact, $A_1 \cap A_2 = \{2, 0, 2\} \cap \{4, 0, 4\}$, so the intersection of the first two sets is already just $\{0\}$.)

Answer 7. $\bigcup_{i\in\mathbb{N}}[0,i+1]=[0,\infty)$, since every non-negative real number is in one of the sets, and the sets contains only non-negative real numbers. And $\bigcap_{i\in\mathbb{N}}[0,i+1]=[0,2]$, since all the sets contain the interval $[0,2]$ and the intersection can't be bigger than $[0,2]$ because $[0,2]$ is one of the sets that is being intersected.

Answer 8. If $\bigcap_{\alpha\in I}A_\alpha=\bigcup_{\alpha\in I}A_\alpha$, then all of the sets A_α must be equal, and each of them is equal to the intersection. Let B be the intersection, which is the same as the union. Let A_α be one of the sets. When you take the union of some sets, every one of those sets is contained in the union, so $A_\alpha\subseteq B$. When you take the intersection of some sets, every one of those sets contains the intersection, so $B\subseteq A_\alpha$. Saying $A_\alpha\subseteq B$ and $B\subseteq A_\alpha$ is the same as saying $A_\alpha=B$. That is, every one of the sets A_α is equal to B.

Answer 9. Yes. If $J\neq\varnothing$ and $J\subseteq I$, then $\bigcap_{\alpha\in I}A_\alpha\subseteq\bigcap_{\alpha\in J}A_\alpha$. If x is in the first intersection, then $x\in A_\alpha$ for every $\alpha\in I$. But since $J\subset I$, it follows that $x\in A_\alpha$ for every $\alpha\in J$, and that means that x is in the second intersection. (We only need $J\neq\varnothing$ because the intersection of zero sets has not been defined.)

Answer 10. If $C=\varnothing$, then $X\times C=\varnothing$ for any set X. So, for example $\{1\}\times\varnothing=\{2\}\times\varnothing$ even though $\{1\}\neq\{2\}$. However, if $A\times C=B\times C$ and $C\neq\varnothing$, then A must equal B. To see this, note that if C is not empty, then there exists some element $c\in C$. Then for any $a\in A$, we have $(a,c)\in A\times C$. But $A\times C=B\times C$, so we also have $(a,c)\in B$, which means that a must be in B also. (Another way of saying this: If $C\neq\varnothing$, then A is clearly equal to the set of first coordinates of ordered pairs in $A\times C$, and B is equal to the set of first coordinates of ordered pairs in $B\times C$. So $A\times C=B\times C$ will imply $A=B$.)