1. Note that $g \circ f : A \to C$, so the first coordinates of the ordered pairs in $g \circ f$, considered as a set of ordered pairs, are $a, b, c, d,$ and $e$. To compute the second coordinates, note, for example, that $g \circ f(a) = g(f(a)) = g(3) = R$, and therefore $(a, R) \in g \circ f$. Doing a similar computation for each element of $A$, we see that

$$g \circ f = \{(a, R), (b, R), (c, R), (d, B), (e, G)\}$$

2. Note, for example, that $h \circ h(a) = h(h(a)) = h(c) = e$, and $h \circ h \circ h(a) = h(h(h(a))) = h(e) = f$. In particular, if we know $h \circ h$, we can use that to compute $h \circ h \circ h$ in one step by forming the composition of $h$ with $h \circ h$.

   a) $h \circ h = \{(a, c), (b, e), (c, f), (d, f), (e, f), (f, f)\} \quad \text{and} \quad h \circ h \circ h = \{(a, f), (b, f), (c, f), (d, f), (e, f), (f, f)\} \quad \text{for all} \ x \in A, \ \text{and} \ \ h(f) = f,$

   b) $h \circ h \circ h = \{(a, e), (b, f), (c, f), (d, f), (e, f), (f, f)\}$

   c) $h \circ h \circ h \circ h = \{(a, f), (b, f), (c, f), (d, f), (e, f), (f, f)\}$

3. The function $f$ is injective. Proof: Suppose that $f(n) = f(m)$. We want to show that $n = m$. Saying $f(n) = f(m)$ means that $(2n, n+3) = (2m, m+3)$. In particular, by definition of equality of ordered pairs, this means that $2n = 2m$. Dividing both sides of this equation by 2 shows that $n = m$.

   However, $f$ is not surjective. Proof: For example, the element $(1,0)$ cannot be equal to $f(n)$ for any $n$ since the first coordinate of $f(n)$ is always an even number.

4. The function $\theta$ is injective: Suppose that $\theta(X) = \theta(Y)$; that is, $X = Y$. Taking the complement of both sides of the equation gives $\overline{X} = \overline{Y}$, and because $\overline{X} = X$ for any set $X$, this means that $X = Y$. The function $\theta$ is also surjective. Given $Y \in \mathcal{P}(A)$, let $X = \overline{Y}$. Then $\theta(X) = \overline{Y} = Y$, so we have shown that every $Y \in \mathcal{P}(A)$ is in the range of $\theta$. Because $\theta$ is both injective and surjective, it is by definition bijective.

5. We first show that $f$ is injective: Suppose that $n, m \in \mathbb{N}$ and $f(n) = f(m)$. We must show that $n = m$. Since $f(n) = f(m)$, we have that $
frac{1}{4}((-1)^n(2n - 1) + 1) = \nfrac{1}{4}((-1)^m(2m - 1) + 1)$. Multiplying by 4 and subtracting 1 from both sides gives $(-1)^n(2n - 1) = (-1)^m(2m - 1)$. Note that $(-1)^n$ is 1 if $n$ is even and is $-1$ if $n$ is odd. The two sides of the equation $(-1)^n(2n - 1) = (-1)^m(2m - 1)$ must have the same sign or must both be zero. If both sides are zero, we must have $2n - 1 = 2m - 1 = 0$; if not, then we see that $(-1)^n$ must equal $(-1)^m$, and we can divide the equation by that number to get $2n - 1 = 2m - 1$. In any case, $2n - 1 = 2m - 1$. Finally, adding 1 and dividing by 2 yields $n = m$.

   Next, we show that $f$ is surjective: Suppose that $k \in \mathbb{Z}$. We must find an $n \in \mathbb{N}$ such that $f(n) = k$, that is, $\nfrac{1}{4}((-1)^n(2n - 1) + 1) = k$. Consider three cases: $k = 0$, $k > 0$, and $k < 0$. For the case $k = 0$ we see that $f(1) = \nfrac{1}{4}((-1)^1(1) + 1) = \nfrac{1}{4}((-1)(1-1)) = 0$. For the case $k > 0$, let $n = 2k$. Noting that $(-1)^{2k} = 1$, we see that $f(n) = f(2k) = \nfrac{1}{4}((-1)^{2k}(2k - 1) + 1) = \nfrac{1}{4}(1 \cdot (4k - 1) + 1) = \nfrac{1}{4}(4k) = k$. For the case $k < 0$, let $n = 1 - 2k$. Noting that $1 - 2k$ is an odd positive number, we see that $f(n) = f(1 - 2k) = \nfrac{1}{4}((-1)^{1-2k}(2(1 - 2k) - 1) + 1) = \nfrac{1}{4}((-1) \cdot (2 - 4k - 1) + 1) = \nfrac{1}{4}((-1 - 4k + 1) = k$. So, in any case, $k$ is in the range of $f$.

   We have shown that $f$ is injective and surjective. Therefore, by definition, it is bijective.
6. Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \).

a) Suppose that \( g \circ f \) is surjective. We want to show that \( g \) is surjective. Let \( c \in C \). We must find a \( b \in B \) such that \( g(b) = c \). Since the function \( g \circ f: A \rightarrow C \) is surjective by assumption, there is an \( a \in A \) such that \( g \circ f(a) = c \). Let \( b = f(a) \). Then \( g(b) = g(f(a)) = g \circ f(a) = c \).

b) We must find an example were \( g \circ f \) is surjective, but \( f \) is not surjective. For the most trivial possible example, let \( A = \{a\} \), \( B = \{1, 2\} \), and \( C = \{c\} \). Define \( f: A \rightarrow B \) by setting \( f(a) = 1 \), and define \( g: B \rightarrow C \) by setting \( g(1) = g(2) = c \). Then \( g \circ f(a) = c \), so \( g \circ f \) is surjective, but \( f \) is not surjective because \( 2 \) is not in the range of \( f \). (For an example with formulas, defining \( f: \mathbb{N} \rightarrow \mathbb{Z} \) by letting \( f(n) = n - 1 \) for all \( n \), and define \( g: \mathbb{Z} \rightarrow \mathbb{N} \) by \( g(m) = 1 + |m| \) for all \( m \in \mathbb{Z} \). Then for \( n \in \mathbb{N} \), \( g \circ f(n) = 1 + |n - 1| = n \), so \( g \circ f \) is surjective. However, \( f \) is not surjective.)

7. Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \).

a) Suppose that \( g \circ f \) is injective. We want to show that \( f \) is injective. Suppose that \( a_1 \) and \( a_2 \) are elements of \( A \) and that \( f(a_1) = f(a_2) \). We must show that \( a_1 = a_2 \). Applying \( g \) to both sides of the equation \( f(a_1) = f(a_2) \), we get that \( g(f(a_1)) = g(f(a_2)) \), that is \( g \circ f(a_1) = g \circ f(a_2) \). Because \( g \circ f \) is injective by assumption, we can conclude that \( a_1 = a_2 \).

b) We must find an example were \( g \circ f \) is injective, but \( g \) is not injective. But in fact, both examples given for the previous problem work here as well.

8. Let \( y \in \mathbb{R} \setminus \{5\} \). To find \( f^{-1}(y) \), we must solve \( f(x) = y \) for \( x \). That is, we must find \( x \in \mathbb{R} \setminus \{2\} \) such that \( \frac{5x + 1}{x - 2} = y \). Multiplying the equation by \( x - 2 \) gives \( 5x + 1 = (x - 2)y = xy - 2y \). Subtracting \( xy \) from both sides gives \( 5x - xy + 1 = -2y \), and subtracting 1 from both sides gives \( 5x - xy = -2y - 1 \), or \( x(5 - y) = -(2y + 1) \). Because \( y \in \mathbb{R} \setminus \{5\} \), we know \( y \neq 5 \) and therefore \( 5 - y \neq 0 \). So we can divide the equation by \( 5 - y \) to give \( y = \frac{2y + 1}{y - 5} \). This computation shows that \( f \left( \frac{2y + 1}{y - 5} \right) = y \) and therefore \( f^{-1}(y) = \frac{2y + 1}{y - 5} \).

We can also check this:

\[
\begin{align*}
    f \left( \frac{2y + 1}{y - 5} \right) &= \frac{5 \left( \frac{2y + 1}{y - 5} \right) + 1}{\frac{2y + 1}{y - 5} - 2} \\
    &= \frac{5(2y + 1) + 1(y - 5)}{(2y + 1) - 2(y - 5)} \\
    &= \frac{10y + 5 + y - 5}{2y + 1 - 2y + 10} \\
    &= \frac{11y}{11} \\
    &= y
\end{align*}
\]

9. We must show that \( s \circ s \) is the identity function on \( \mathbb{N} \). We note that if \( n \) is even then \( s(n) = n + 1 \) is odd, and that if \( n \) is odd then \( s(n) = n - 1 \) is even. It follows that in the case when \( n \) is even, \( s \circ s(n) = s(n + 1) = (n + 1) - 1 = n \), and in the case where \( n \) is odd, \( s \circ s(n) = s(n - 1) = (n - 1) + 1 = n \). So, in any case, \( s \circ s(n) = n \), which means that \( s = s^{-1} \).

Another function that is its own inverse is \( \nu: \mathbb{Z} \rightarrow \mathbb{Z} \) defined as \( \nu(n) = -1 \). This function is its own inverse because \( \nu \circ \nu(n) = \nu(-n) = -(-n) = n \) for all \( n \in \mathbb{Z} \).