Math 135, Fall 2019, Sample Answers to Homework 10

1. Note that $g \circ f \colon A \to C$, so the first coordinates of the ordered pairs in $g \circ f$, considered as a set of ordered pairs, are a, b, c, d, and e. To compute the second coordinates, note, for example, that $g \circ f(a) = g(f(a)) = g(3) = R$, and therefore $(a, R) \in g \circ f$. Doing a similar computation for each element of A, we see that

$$g \circ f = \{(a, R), (b, R), (c, R), (d, B), (e, G)\}$$

2. Note, for example, that $h \circ h(a) = h(h(a)) = h(c) = e$, and $h \circ h \circ h(a) = h(h \circ h(a)) = h(e) = f$. In particular, if we know $h \circ h$, we can use that to compute $h \circ h \circ h$ in one step by forming the composition of h with $h \circ h$.

a) $h \circ h = \{(a, c), (b, e), (c, f), (d, f), (e, f), (f, f)\}$

b) $h \circ h \circ h = \{(a, e), (b, f), (c, f), (d, f), (e, f), (f, f)\}$

c) $h \circ h \circ h \circ h = \{(a, f), (b, f), (c, f), (d, f), (e, f), (f, f)\}$

d) Since $(h \circ h \circ h \circ h)(x) = f$ for all $x \in A$, and $h(f) = f$, As we continue to add on additional compositions with h, we will always get a function whose value is f for all $x \in A$.

3. The function f is injective. Proof: Suppose that $f(n) = f(m)$. We want to show that $n = m$. Saying $f(n) = f(m)$ means that $(2n, n+3) = (2m, m+3)$. In particular, by definition of equality of ordered pairs, this means that $2n = 2m$. Dividing both sides of this equation by 2 shows that $n = m$.

Howeve, f is not surjective. Proof: For example, the element $(1, 0)$ cannot be equal to $f(n)$ for any n since the first coordinate of $f(n)$ is always an even number.

4. The function θ is injective: Suppose that $\theta(X) = \theta(Y)$; that is, $\overline{X} = \overline{Y}$. Taking the complement of both sides of the equation gives $\overline{\overline{X}} = \overline{\overline{Y}}$, and because $\overline{\overline{X}} = X$ for any set X, this means that $X = Y$. The function θ is also surjective. Given $Y \in \mathscr{P}(A)$, let $X = \overline{Y}$. Then $\theta(X) = \overline{\overline{Y}} = Y$, so we have shown that every $Y \in \mathscr{P}(A)$ is in the range of θ. Because θ is both injective and surjective, it is by definition bijective.

5. We first show that f is injective: Suppose that $n, m \in \mathbb{N}$ and $f(n) = f(m)$. We must show that $n = m$. Since $f(n) = f(m)$, we have that $\frac{1}{4}\big((-1)^n(2n-1)+1\big) = \frac{1}{4}\big((-1)^m(2m-1)+1\big)$. Multiplying by 4 and subtracting 1 from both sides gives $(-1)^n(2n-1) = (-1)^m(2m-1)$. Note that $(-1)^n$ is 1 if n is even and is -1 if n is odd. The two sides of the equation $(-1)^n(2n-1) = (-1)^m(2m-1)$ must have the same sign or must both be zero. If both sides are zero, we must have $2n-1 = 2m-1 = 0$; if not, then we see that $(-1)^n$ must equal $(-1)^m$, and we can divide the equation by that number to get $2n-1 = 2m-1$. In any case, $2n-1 = 2m-1$. Finally, adding 1 and dividing by 2 yields $n = m$.

Next, we show that f is surjective: Suppose that $k \in \mathbb{Z}$. We must find an $n \in \mathbb{N}$ such that $f(n) = k$, that is, $\frac{1}{4}\big((-1)^n(2n-1)+1\big) = k$. Consider three cases: $k = 0$, $k > 0$, and $k < 0$. For the case $k = 0$ we see that $f(1) = \frac{1}{4}\big((-1)^1(2\cdot 1-1)+1\big) = \frac{1}{4}\big((-1)(1-1)\big) = 0$. For the case $k > 0$, let $n = 2k$. Noting that $(-1)^{2k} = 1$, we see that $f(n) = f(2k) = \frac{1}{4}\big((-1)^{2k}(2(2k)-1)+1\big) = \frac{1}{4}(1 \cdot (4k-1)+1) = \frac{1}{4}(4k) = k$. For the case $k < 0$, let $n = 1-2k$. Noting that $1-2k$ is an odd positive number, we see that $f(n) = f(1-2k) = \frac{1}{4}\big((-1)^{1-2k}(2(1-2k)-1)+1\big) = \frac{1}{4}((-1)\cdot(2-4k-1)+1) = \frac{1}{4}(-(1-4k)+1) = k$. So, in any case, k is in the range of f.

We have shown that f is injective and surjective. Therefore, by definition, it is bijective.

6. Let $f\colon A \to B$ and $g\colon B \to C$.

a) Suppose that $g \circ f$ is surjective. We want to show that g is surjective. Let $c \in C$. We must find a $b \in B$ such that $g(b) = c$. Since the function $g \circ f\colon A \to C$ is surjective by assumption, there is an $a \in a$ such that $g \circ g(a) = c$. Let $b = f(a)$. Then $g(b) = g(f(a)) = g \circ f(a) = c$.

b) We must find an example were $g \circ f$ is surjective, but f is not surjective. For the most trivial possible example, let $A = \{a\}$, $B = \{1, 2\}$, and $C = \{c\}$. Define $f\colon A \to B$ by setting $f(a) = 1$, and define $g\colon B \to C$ by setting $g(1) = g(2) = c$. Then $g \circ f(a) = c$, so $g \circ f$ is surjective, but f is not surjective because 2 is not in the range of f. (For an example with formulas, defining $f\colon \mathbb{N} \to \mathbb{Z}$ by letting $f(n) = n - 1$ for all n, and define $g\colon \mathbb{Z} \to \mathbb{N}$ by $g(m) = 1 + |m|$ for all $m \in \mathbb{Z}$. Then for $n \in \mathbb{N}$, $g \circ f(n) = 1 + |n - 1| = n$, so $g \circ f$ is surjective. However, f is not surjective.)

7. Let $f\colon A \to B$ and $g\colon B \to C$.

a) Suppose that $g \circ f$ is injective. We want to show that f is surjective. Suppose that a_1 and a_2 are elements of A and that $f(a_1) = f(a_2)$. We must show that $a_1 = a_2$. Applying g to both sides of the equation $f(a_1) = f(a_2)$, we get that $g(f(a_1)) = g(f(a_2))$, that is $g \circ f(a_1) = g \circ f(a_2)$. Because $g \circ f$ is injective by assumption, we can conclude that $a_1 = a_2$.

b) We must find an example were $g \circ f$ is injective, but g is not injective. But in fact, both examples given for the previous problem work here as well.

8. Let $y \in \mathbb{R} \smallsetminus \{5\}$. To find $f^{-1}(y)$, we must solve $f(x) = y$ for x. That is, we must find $x \in R \smallsetminus \{2\}$ such that $\frac{5x+1}{x-2} = y$. Multiplying the equation by $x - 2$ gives $5x + 1 = (x - 2)y = xy - 2y$. Subtract xy from both sides gives $5x - xy + 1 = -2y$, and subtracting 1 from both sides gives $5x - xy = -2y - 1$, or $x(5 - y) = -(2y + 1)$. Because $y \in \mathbb{R} \smallsetminus \{5\}$, we know $y \neq 5$ and therefore $5 - y \neq 0$. So we can divide the equation by $5 - y$ to give $y = \frac{-(2y+1)}{5-y} = \frac{2y+1}{y-5}$. This computation shows that $f\big(\frac{2y+1}{y-5}\big) = y$ and therefore $f^{-1}(y) = \frac{2y+1}{y-5}$.

We can also check this:

$$\begin{aligned}
f\left(\frac{2y+1}{y-5}\right) &= \frac{5\big(\frac{2y+1}{y-5}\big) + 1}{\frac{2y+1}{y-5} - 2} \\
&= \frac{5(2y+1) + 1(y-5)}{(2y+1) - 2(y-5)} \\
&= \frac{10y + 5 + y - 5}{2y + 1 - 2y + 10} \\
&= \frac{11y}{11} \\
&= y
\end{aligned}$$

9. We must show that $s \circ s$ is the identity function on $\mathbb{N}$. We note that if n is even then $s(n) = n + 1$ is odd, and that if n is odd then $s(n) = n - 1$ is even. It follows that in the case when n is even, $s \circ s(n) = s(n + 1) = (n + 1) - 1 = n$, and in the case where n is odd, $s \circ s(n) = s(n - 1) = (n - 1) + 1 = n$. So, in any case, $s \circ s(n) = n$, which means that $s = s^{-1}$.

Another function that is its own inverse is $\nu\colon \mathbb{Z} \to \mathbb{Z}$ defined as $\nu(n) = -1$. this function is its own inverse because $\nu \circ \nu(n) = \nu(-n) = -(-n) = n$. for all $n \in \mathbb{Z}$,