Math 135, Fall 2019, Homework 2

This homework is due next Monday, September 9

Exercise 1. Consider the set of real numbers, $\mathbb{R}$, with its usual topology, and consider the subset $[1,7) = \{x \in \mathbb{R} : 1 \leq x < 7\}$. Show that the set $[1,7)$ is *not closed* in $\mathbb{R}$. (Hint: Consider the point 1, which is an element of $[1,7)$.) Now show that the set $[1,7)$ is *not open.* (Hint: It is enough to show that the complement of $[1,7)$ is not closed.)

Exercise 2 (Exercise 1 from the handout)**.** Explain why a subset of $\mathbb{R}$ that consists of a single element, such as $\{2\}$, is a closed subset of $\mathbb{R}$. Based on that fact and properties of closed sets, explain why every finite subset of $\mathbb{R}$ is a closed subset of $\mathbb{R}$.

Exercise 3. Let A be the two-element set $A = \{a, b\}$. Show that each of the following collections of subsets of A is a topology for A:

a) $\mathcal{T} = \{\varnothing, \{a,b\}\}$
b) $\mathcal{S} = \{\varnothing, \{a\}, \{a,b\}\}$
c) $\mathcal{R} = \{\varnothing, \{b\}, \{a,b\}\}$
d) $\mathcal{V} = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$

Now explain why $\mathcal{T}$, $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{V}$ are the **only** possible topologies on the set A.

Exercise 4 (Exercise 2 from the handout)**.** Let X be any set. Let $\mathcal{T}$ be the the collection of **all** subsets of X (that is, $\mathcal{T} = \mathscr{P}(X)$). Explain why $\mathcal{T}$ is a topology for X. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X, and that the sequence converges to x. Show that there is a natural number N such that $x_n = x$ for all $n \geq N$. (Hint: For $x \in X$, the set $\{x\}$ is an open set that contains X.)

Exercise 5. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. We define the **closure**, A^c, of A to be the smallest closed subset of X that contains A. That is, A^c satisfies the properties **(a)** $A \subseteq A^c$; **(b)** A^c is closed; and $A^c \subseteq F$ for any closed subset F for which $A \subseteq F$.

Let $\mathscr{F}$ be the collection of all possible closed subsets of X that contain A as a subset: $\mathscr{F} = \{F \subseteq X : F \text{ is closed and } A \subseteq F\}$. Show that if we define $A^c = \bigcap_{F \in \mathscr{F}} F$, then A^c satisfies the three properties of the closure of A, as given above.

Finally, in $\mathbb{R}$ with its usual topology, what do you think is the closure of the open interval $(0,1)$? (For this part of the problem, you are not asked to justify your answer.)

Exercise 6 (Exercise 7 from the handout)**.** Let $X = \mathbb{N} \cup \{0\}$. That is, X is the set of non-negative integers. In this exercise, we consider X with a rather strange topology. For each $i = 0, 1, 2, 3, \dots$, define N_i to be the set $N_i = \{0, 1, 2, \dots, i-1\}$. So $N_o = \varnothing$, $N_1 = \{0\}$, $N_2 = \{0, 1\}$, $N_3 = \{0, 1, 2\}$, and so on. We can define a topology on X in which the open sets are precisely the sets $N_o, N_1, N_2, \dots$, together with X itself. That is, we make the topological space $(X, \mathcal{T})$ where $\mathcal{T} = \{X, N_o, N_1, N_2, N_3, \dots\}$. Show that $\mathcal{T}$ is in fact a topology. (What is the union of a family of sets in $\mathcal{T}$? What is the intersection of a family of sets in $\mathcal{T}$?) What are the closed subsets in this topological space?