Math 135, Fall 2019, Sample Answers to Homework 7

7.2 Suppose that x is an odd integer. We must show that $3x+6$ is odd. Sinde x is odd, $x = 3k+1$ for some integer k. Then, $3k+6 = 3(2k+1)+6 = 6k+3+6 = 6k+8+1 = 2(3k+4)+1$. Since $3k+4$ is an integer, $3k+6$ is odd.

Conversely, we must prove that if $3x+6$ is odd, then x is odd. We prove the contrapositive: If x is even, then $3x+6$ is even. Suppose that x is an even integer. Then $x = 2k$ for some integer k, and $3x+6 = 6k+6 = 2(3k+3)$. Since $3k+3$ is an integer, this shows that $3x+6$ is even.

7.8 Suppose that a and b are integers, and suppose that $a \equiv b \pmod{10}$. We must show that $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$. Since $a \equiv b \pmod{10}$, we know that $10 \mid (a-b)$. Since $2 \mid 10$ and $10 \mid (a-b)$, we know from a previous result that $2 \mid (a-b)$, which means $a \equiv b \pmod{2}$. Similarly, since $5 \mid 10$, $a \equiv b \pmod{10}$.

Conversely, we must show that if $a \equiv b \pmod{2}$ and $a \equiv b \pmod{5}$, then $a \equiv (a-b) \pmod{10}$. Since $a \equiv b \pmod{2}$, we have that $2 \mid (a-b)$, and $a-b = 2k$ for some integer k. This means that $a-b$ is even. Similarly, $a-b = 5\ell$ for some integer ℓ. Since $a-b$ is even and 5 is odd, ℓ must be even (because if ℓ were odd, the product of 5 and ℓ would be odd). So $\ell = 2j$ for some integer j, and $a-b = 5 \cdot 2j = 10j$. This shows $10 \mid (a-b)$, which means that $a \equiv b \pmod{10}$.

7.12 Let $x = 1/4$. Then $x^2 = 1/16$ and $\sqrt{x} = 1/2$, so $x^2 < \sqrt{x}$.

7.16 Suppose that a and b are integers and ab is odd. Then both a and b must be odd, since if one of them were even, then their product ab would be even. We know that the square of an odd number is odd, so a^2 and b^2 are both odd. We know that the sum of two odd numbers is even, so a^2+b^2 is even.

7.32 Let $n \in \mathbb{Z}$. We want to show that $gcd(n, n+2)$ is either 1 or 2. Let $d = gcd(a,b)$. Then d is a common divisor of n and $n+2$, so $d|n$ and $d|(n+2)$. This implies that n divides the difference, $(n+2)-2$, which is 2. So d is a divisor of 2. The only divisors of 2 are ± 1 and ± 2. Since a greatest common divisor is positive, d is either 1 or 2.

8.4 Let $a \in \{x \in \mathbb{Z} : mn \mid x\}$. We want to show $a \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$. Since $mn \mid a$ and $m \mid mn$, it follows that $m \mid a$. This means $a \in \{x \in \mathbb{Z} : m \mid x\}$. Similarly, since $n|mn$, $a \in \{x \in \mathbb{Z} : n \mid x\}$. Because a is an element of both $a \in \{x \in \mathbb{Z} : m \mid x\}$ and $a \in \{x \in \mathbb{Z} : n \mid x\}$, it is by definition an element of their intersection.

8.10 For any $x \in U$, we must show that $x \in \overline{A \cap B}$ if and only if $x \in \overline{A} \cup \overline{B}$. But for any such x, we have

$$\begin{aligned}
x \in \overline{A \cap B} &\Leftrightarrow \sim (x \in A \cap B) \\
&\Leftrightarrow \sim (x \in A \wedge x \in B) \\
&\Leftrightarrow (\sim (x \in A)) \vee (\sim (x \in B)) \\
&\Leftrightarrow x \in \overline{A} \vee x \in \overline{B} \\
&\Leftrightarrow x \in \overline{A} \cup \overline{B}
\end{aligned}$$

8.16 The elements of the sets in question are ordered pairs. We must show that for any pair (x, y), $(x, y) \in A \times (B \cup C) \Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$. But for any x,

$$\begin{aligned} (x, y) \in A \times (B \cup C) &\Leftrightarrow (x \in A) \wedge (y \in B \cup C) \\ &\Leftrightarrow (x \in A) \wedge (y \in B \vee y \in C) \\ &\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in B) \\ &\Leftrightarrow ((x, y) \in A \times B) \vee ((x, y) \in A \times B) \\ &\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C) \end{aligned}$$

8.22 Let A and B be sets and assume that $A \subseteq B$. We want to show $A \cap B = A$. For any sets A and B, it is true by the definition of intersection that $A \cap B \subseteq A$, so we only need to show that $A \subseteq A \cap B$. Let $a \in A$. Since $A \subseteq B$, it is also true that $a \in B$. Since $a \in A$ and $a \in B$, then $a \in A \cap B$. We have shown that element of A is an element of $A \cap B$; that is, $A \subseteq A \cap B$.

Conversely, assume that $A \cap B = A$. We want to show that $A \subseteq B$. Let $a \in A$. Since $A \cap B = A$, this means that $a \in A \cap B$. By definition of intersection, it follows that $a \in B$. We have shown every element of A is an element of B; that is, $A \subseteq B$.