Exercise 10.1. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

Proof. We use proof by induction. For the base case, $n = 1$, the formula becomes $1^2 = \frac{1(2)(3)}{6}$, which is true.

For the inductive case, let $k \in \mathbb{N}$, and suppose that we already know that $1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$. We must that that the formula also holds for $k+1$: $1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$. The left hand side of this formula can be written

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\
&= \left(\frac{k(k+1)(2k+1)}{6}\right) + (k+1)^2 \\
&= \left(\frac{2k^3 + 3k^2 + k}{6}\right) + (k^2 + 2k + 1) \\
&= \frac{2k^3 + 3k^2 + k + 6k^2 + 12k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

while the right hand side can be written

$$\begin{aligned}
\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} &= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k^2+3k+2)(2k+3)}{6} \\
&= \frac{2k^3 + 6k^2 + 4k + 3k^2 + 9k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

Since the two sides of the formula are equal, we have proved that it holds for $k+1$. □

Exercise 10.8. Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all $n \in N$.

Proof. We use proof by induction. For the base case, $n = 1$, the formula becomes $\frac{1}{2!} = 1 - \frac{1}{(1+1)!}$. This is equivalent to $\frac{1}{2} = 1 - \frac{1}{2}$, which is true.

For the inductive case, let $k \in \mathbb{N}$, and suppose that we already know that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. We must show that the same formula holds for $k+1$. But

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k+1}{((k+1)+1)!} &= \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{k+1}{(k+2)!} \\
&= \left(1 - \frac{1}{(k+1)!}\right) + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)\cdot(k+1)!}
\end{aligned}$$

$$
\begin{aligned}
&= 1 - \frac{k+2}{(k+2)\cdot(k+1)!} + \frac{k+1}{(k+2)\cdot(k+1)!} \\
&= 1 + \frac{-(k+2)+(k+1)}{(k+2)\cdot(k+1)!} \\
&= 1 + \frac{-1}{(k+2)\cdot(k+1)!} \\
&= 1 - \frac{1}{(k+2)!} \\
&= 1 - \frac{1}{((k+1)+1)!}
\end{aligned}
$$

so the formula holds for $k+1$. □

Exercise 10.10. Prove that $3 \mid (5^{2n} - 1)$ for every integer $n > 0$.

Proof. We use proof by induction. For the base case, $n = 0$, we musth show that $3 \mid (5^0 - 1)$. Since $5^0 = 1$, this is equivalent to $3|(1-1)$, which is true because every non-zero integer divides 0.

For the inductive case, let $k = 0$, and suppose that $3 \mid (5^{2k} - 1)$. We must show that $3 \mid (5^{2(k+1)} - 1)$. But

$$
\begin{aligned}
5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\
&= (5^{2k} \cdot 5^2) - 1 \\
&= (25 \cdot 5^{2k}) - 1 \\
&= (25 \cdot 5^{2k} - 25) + 25 - 1 \\
&= 25(5^{2k} - 1) + 24.
\end{aligned}
$$

Since $3 \mid (5^{2k} - 1)$ by the induction hypothesis and $3 \mid 24$, it follows that $3 \mid (25(5^{2k} - 1) + 24)$. That is, $3 \mid (5^{2(k+1)} - 1)$. So the theorem holds for $k+1$. □

Exercise 10.18. We consider subsets of some universal set U. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$ for all $n \geq 2$ and all subsets $A_1, A_2, \ldots, A_n$ of U.

Proof. We use proof by induction.

Base Case, $n = 2$: We want to show $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ for all subsets A_1 and A_2 of U. But this is just DeMorgan's law for sets, which we have already proved.

Inductive Case. Let $k \geq 2$ and suppose we already know that $\overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$ for any k subsets of U. Consider any $k+1$ subsets $A_1, A_2, \ldots A_{k+1}$. Then we have

$$
\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup \cdots A_k) \cup A_{k+1}} \\
&= \overline{A_1 \cup A_2 \cup \cdots A_k} \cap \overline{A_{k+1}} && \text{by the } n = 2 \text{ case} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}} && \text{by the inductive hypothesis}
\end{aligned}
$$

so the theorem is true for any $k+1$ subsets of U. □

Exercise 10.34. Prove that $3^1 + 3^2 + 3^3 + \cdots + 3^n = \frac{3^{n+1}-3}{2}$ for every $n \in \mathbb{N}$.

Proof. We use proof by induction.

Base Case, $n = 1$: For $n = 1$, the statement becomes $3^1 = \frac{3^2-3}{2}$. Since $\frac{3^2-3}{2} = \frac{9-3}{2} = \frac{6}{2} = 3$, the statement is true for $n = 1$.

Inductive case. Let $k \geq 1$, and suppose that $3^1 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1}-3}{2}$. We must show that $3^1 + 3^2 + 3^3 + \cdots + 3^{k+1} = \frac{3^{k+2}-3}{2}$ But

$$\begin{aligned}
3^1 + 3^2 + 3^3 + \cdots + 3^{k+1} &= \left(3^1 + 3^2 + 3^3 + \cdots + 3^k\right) + 3^{k+1} \\
&= \left(\frac{3^{k+1}-3}{2}\right) + 3^{k+1} \\
&= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} \\
&= \frac{3 \cdot 3^{k+1} - 3}{2} \\
&= \frac{3^{k+2} - 3}{2}
\end{aligned}$$

so the statement is true for $n = k + 1$. □

Extra Exercise 1. Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

(a) Proof by induction. Base Case: When $n = 1$, the statement becomes $\frac{1}{1(1+1)} = \frac{1}{1+1}$, so the statement is true for $n = 1$.

Inductive case: Let $k \in \mathbb{N}$, and suppose that $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. we must show $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$. But

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+1+1)} \\
&= \left(\frac{k}{k+1}\right) + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

so the statement is true for $n = k + 1$.

(a) Direct proof. Note that $\frac{1}{i} - \frac{1}{i+1} = \frac{(i+1)-i}{i(i+1)} = \frac{1}{i(i+1)}$, so we can write

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$

$$
\begin{aligned}
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1)}\right) \\
&= \frac{1}{1} - \frac{1}{n+1} \\
&= \frac{(n+1)-1}{n+1} \\
&= \frac{n}{n+1}
\end{aligned}
$$

Extra Exercise 1. Prove that $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ for all integers $n \geq 0$.

(a) Proof by induction. Base Case: When $n = 0$, the statement becomes $r^0 = \frac{1-r^1}{1-r}$, which reduces to $1 = 1$. So the statement is true for $n = 1$.

Inductive Case: Let $k \geq 0$, and suppose that $\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$. We must show $\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r}$ But

$$
\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \left(\sum_{i=0}^{k} r^i\right) + r^{k+1} \\
&= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\
&= \frac{1-r^{k+1}+(1-r)r^{k+1}}{1-r} \\
&= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\
&= \frac{1-r^{k+2}}{1-r}
\end{aligned}
$$

so the statement holds for $n = k+1$.

(a) Direct proof. Let $S = \sum_{i=0}^{n} r^i = 1+r+r^2+r^3+\cdots+r^n$. Then $rS = r+r^2+r^3+r^4\cdots+r^{n+1}$, and $S - rS = 1 - r^{n+1}$. Factoring $S - rS = S(1-r)$ and dividing by $1-r$ gives $S = \frac{1-r^{n+1}}{1-r}$, as we wanted to show.