Exercise 10.1. Prove that \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\) for all \(n \in \mathbb{N}\).

Proof. We use proof by induction. For the base case, \(n = 1\), the formula becomes \(1^2 = \frac{1(2)(3)}{6}\), which is true.

For the inductive case, let \(k \in \mathbb{N}\), and suppose that we already know that \(1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}\). We must show that the formula also holds for \(k + 1\): \(1^2 + 2^2 + 3^2 + \cdots + (k + 1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}\). The left hand side of this formula can be written

\[
1^2 + 2^2 + 3^2 + \cdots + (k + 1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k + 1)^2
\]

\[
= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2
\]

\[
= \frac{2k^3 + 3k^2 + k}{6} + (k^2 + 2k + 1)
\]

\[
= \frac{2k^3 + 3k^2 + k + 6k^2 + 12k + 6}{6}
\]

\[
= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\]

while the right hand side can be written

\[
\frac{(k + 1)(k + 2)(2(k + 1) + 1)}{6} = \frac{(k + 1)(k + 2)(2k + 3)}{6}
\]

\[
= \frac{(k^2 + 3k + 2)(2k + 3)}{6}
\]

\[
= \frac{2k^3 + 6k^2 + 4k + 3k^2 + 9k + 6}{6}
\]

\[
= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\]

Since the two sides of the formula are equal, we have proved that it holds for \(k + 1\). \(\square\)

Exercise 10.8. Prove \(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}\) for all \(n \in \mathbb{N}\).

Proof. We use proof by induction. For the base case, \(n = 1\), the formula becomes \(\frac{1}{2!} = 1 - \frac{1}{2!}\). This is equivalent to \(\frac{1}{2} = 1 - \frac{1}{2}\), which is true.

For the inductive case, let \(k \in \mathbb{N}\), and suppose that we already know that \(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}\). We must show that the same formula holds for \(k + 1\). But

\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k + 1}{((k+1) + 1)!} = \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{k + 1}{(k+2)!}
\]

\[
= \left(1 - \frac{1}{(k+1)!}\right) + \frac{k + 1}{(k+2)!}
\]

\[
= 1 - \frac{1}{(k+1)!} + \frac{k + 1}{(k+2) \cdot (k + 1)!}
\]
\[= 1 - \frac{k + 2}{(k + 2) \cdot (k + 1)!} + \frac{k + 1}{(k + 2) \cdot (k + 1)!}\]
\[= 1 + \frac{-1}{(k + 2) \cdot (k + 1)!}\]
\[= 1 - \frac{1}{(k + 2)!}\]
\[= 1 - \frac{1}{((k + 1) + 1)!}\]

so the formula holds for \(k + 1\).

Exercise 10.10. Prove that \(3 \mid (5^{2n} - 1)\) for every integer \(n > 0\).

Proof. We use proof by induction. For the base case, \(n = 0\), we must show that \(3 \mid (5^0 - 1)\). Since \(5^0 = 1\), this is equivalent to \(3 \mid (1 - 1)\), which is true because every non-zero integer divides 0.

For the inductive case, let \(k = 0\), and suppose that \(3 \mid (5^{2k} - 1)\). We must show that \(3 \mid (5^{2(k+1)} - 1)\). But

\[5^{2(k+1)} - 1 = 5^{2k+2} - 1\]
\[= (5^{2k} \cdot 5^2) - 1\]
\[= (25 \cdot 5^{2k}) - 1\]
\[= (25 \cdot 5^{2k} - 25 + 24)\]
\[= 25(5^{2k} - 1) + 24.\]

Since \(3 \mid (5^{2k} - 1)\) by the induction hypothesis and \(3 \mid 24\), it follows that \(3 \mid (25(5^{2k} - 1) + 24)\). That is, \(3 \mid (5^{2(k+1)} - 1)\). So the theorem holds for \(k + 1\).

Exercise 10.18. We consider subsets of some universal set \(U\). Prove that \(A_1 \cup A_2 \cup \cdots \cup A_n = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\) for all \(n \geq 2\) and all subsets \(A_1, A_2, \ldots, A_n\) of \(U\).

Proof. We use proof by induction.

Base Case, \(n = 2\): We want to show \(\overline{A_1} \cap \overline{A_2} = A_1 \cup A_2\) for all subsets \(A_1\) and \(A_2\) of \(U\). But this is just DeMorgan’s law for sets, which we have already proved.

Inductive Case. Let \(k \geq 2\) and suppose we already know that \(\overline{A_1} \cup A_2 \cup \cdots \cup A_k = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}\) for any \(k\) subsets of \(U\). Consider any \(k + 1\) subsets \(A_1, A_2, \ldots, A_{k+1}\). Then we have

\[\overline{A_1} \cup A_2 \cup \cdots \cup A_{k+1} = (A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}\]
\[= A_1 \cup A_2 \cup \cdots \cup A_k \cap \overline{A_{k+1}}\]
\[= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}\]

by the \(n = 2\) case

by the inductive hypothesis

so the theorem is true for any \(k + 1\) subsets of \(U\).
Exercise 10.34. Prove that $3^1 + 3^2 + 3^3 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}$ for every $n \in \mathbb{N}$.

Proof. We use proof by induction.

Base Case, $n = 1$: For $n = 1$, the statement becomes $3^1 = \frac{3^2 - 3}{2}$. Since $\frac{3^2 - 3}{2} = \frac{9 - 3}{2} = \frac{6}{2} = 3$, the statement is true for $n = 1$.

Inductive case. Let $k \geq 1$, and suppose that $3^1 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 3}{2}$. We must show that $3^1 + 3^2 + 3^3 + \cdots + 3^{k+1} = \frac{3^{k+2} - 3}{2}$

But

$$3^1 + 3^2 + 3^3 + \cdots + 3^{k+1} = (3^1 + 3^2 + 3^3 + \cdots + 3^k) + 3^{k+1} = \left(\frac{3^{k+1} - 3}{2}\right) + 3^{k+1} = \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} = \frac{3 \cdot 3^{k+1} - 3}{2} = \frac{3^{k+2} - 3}{2}$$

so the statement is true for $n = k + 1$.

\[\square\]

Extra Exercise 1. Prove that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

(a) Proof by induction. Base Case: When $n = 1$, the statement becomes $\frac{1}{1(1+1)} = \frac{1}{1+1}$, so the statement is true for $n = 1$.

Inductive case: Let $k \in \mathbb{N}$, and suppose that $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. We must show $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.

But

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)(k+2)} = \left(\frac{k}{k+1}\right) + \frac{1}{(k+1)(k+2)} = \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

so the statement is true for $n = k + 1$.

(a) Direct proof. Note that $\frac{1}{i} - \frac{1}{i+1} = \frac{(i+1) - i}{i(i+1)} = \frac{1}{i(i+1)}$, so we can write

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}$$
\[
\begin{align*}
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{n+1} \\
&= \frac{(n+1) - 1}{n+1} \\
&= \frac{n}{n+1}
\end{align*}
\]

Extra Exercise 1. Prove that \( \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1-r} \) for all integers \( n \geq 0 \).

(a) **Proof by induction.** Base Case: When \( n = 0 \), the statement becomes \( r^0 = \frac{1-r^1}{1-r} \), which reduces to \( 1 = 1 \). So the statement is true for \( n = 1 \).

Inductive Case: Let \( k \geq 0 \), and suppose that \( \sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r} \). We must show \( \sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r} \).

But
\[
\begin{align*}
\sum_{i=0}^{k+1} r^i &= \left(\sum_{i=0}^{k} r^i\right) + r^{k+1} \\
&= \frac{1 - r^{k+1}}{1-r} + r^{k+1} \\
&= \frac{1 - r^{k+1} + (1-r)r^{k+1}}{1-r} \\
&= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1-r} \\
&= \frac{1 - r^{k+2}}{1-r}
\end{align*}
\]

so the statement holds for \( n = k + 1 \).

(a) **Direct proof.** Let \( S = \sum_{i=0}^{n} r^i = 1+r+r^2+r^3+\cdots+r^n \). Then \( rS = r+r^2+r^3+r^4+\cdots+r^{n+1} \), and \( S - rS = 1 - r^{n+1} \). Factoring \( S - rS = S(1-r) \) and dividing by \( 1-r \) gives \( S = \frac{1-r^{n+1}}{1-r} \), as we wanted to show.