Problem 1. Produce **three** other matrices that are row equivalent to the following matrix. State what you did to get each new matrix. If you understand what row equivalence means, this exercise is very easy!

$$\begin{pmatrix} 3 & 4 & -2 \\ 1 & 1 & 7 \\ -5 & 3 & 1 \end{pmatrix}$$

Any row operation, or sequence of row operations, applied to the original matrix produces a row-equivalent matrix. Here are matrices produced by $\rho_1 \leftrightarrow \rho_2$, $4\rho_1$, and $5\rho_2 + \rho_3$ all applied to the original matrix:

$$\begin{pmatrix} 1 & 1 & 7 \\ 3 & 4 & -2 \\ -5 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 12 & 16 & -8 \\ 1 & 1 & 7 \\ -5 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 & -2 \\ 1 & 1 & 7 \\ 0 & 8 & 36 \end{pmatrix}$$

Problem 2. Apply row operations to put the matrix on the left below into reduced echelon form. Show the full sequence of operations that you apply. The only correct answer is shown on the right. The point of the problem is to show the computation.

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 1 & 2 & 4 & 2 & 8 \\ -2 & 2 & 10 & 2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 1 & 2 & 4 & 2 & 8 \\ -2 & 2 & 10 & 2 & 2 \end{pmatrix} \xrightarrow[2\rho_1+\rho_4]{-\rho_1+\rho_3} \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 0 & 2 & 6 & 2 & 6 \\ 0 & 2 & 6 & 2 & 6 \end{pmatrix} \xrightarrow{-\rho_3+\rho_4} \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 0 & 2 & 6 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 0 & 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}\rho_3} \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 2 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-2\rho_3+\rho_2} \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 3. Assuming that the matrix in the previous problem represents a **homogeneous** system of linear equations, use the reduced echelon form of the matrix to find the solution set of that system. (The constant terms in the equations, which are all zero, are omitted from the matrix!)

Use x, y, z, w, t as the variable names. Note that z and t are free variables. We can then solve the system as

$$\begin{aligned} t &= b \\ w &= -4t = -4b \\ z &= a \\ y &= -3z + t = -3a + b \\ x &= 2z - 2t = 2a - 2b \end{aligned}$$

The solution set can be written

$$\left\{ \begin{pmatrix} 2a-2b \\ -3a+b \\ a \\ -4b \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 1 \\ 0 \\ -4 \\ 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Problem 4. Put the following eight matrices into groups, so that each matrix is row-equivalent to all the other matrices in the group, but not row-equivalent to matrices from other groups. (Remember how to use reduced echelon form to tell whether two matrices are row equivalent.)

a) $\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$ **d)** $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ **f)** $\begin{pmatrix} \frac{1}{3} & -1 \\ -3 & 9 \end{pmatrix}$ **g)** $\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ **h)** $\begin{pmatrix} 0 & 2 \\ 3 & 7 \end{pmatrix}$

To answer this question, we can reduce each matrix to reduced echelon form. Matrices that have the same reduced echelon form are row equivalent to each other and so are in the same group.

(a) $\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix} \xrightarrow{-1\rho_1} \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}\rho_1} \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \xrightarrow{-\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$\xrightarrow{\frac{1}{3}\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{2}\rho_2+\rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \xrightarrow{-\frac{1}{3}\rho_1 + \rho_2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix} \xrightarrow{\frac{1}{3}\rho_1} \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \xrightarrow{2\rho_1 + \rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow{-\rho_1 + \rho_2} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}\rho_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{2\rho_2 + \rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} \frac{1}{3} & -1 \\ -3 & 9 \end{pmatrix} \xrightarrow{3\rho_1 + \rho_2} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(h) $\begin{pmatrix} 0 & 2 \\ 3 & 7 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 3 & 7 \\ 0 & 2 \end{pmatrix} \xrightarrow[\frac{1}{2}\rho_2]{\frac{1}{3}\rho_1} \begin{pmatrix} 1 & \frac{7}{3} \\ 0 & 1 \end{pmatrix} \xrightarrow{-\frac{3}{7}\rho_2 + \rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So, we see that the groups are: group 1: a and f; group 2: b, e, and h; group 3: c by itself; and group 4: d and g.

Problem 5. Let W be a subset of $\mathbb{R}^2$ that is a vector space using the addition and scalar multiplication operations from $\mathbb{R}^2$. Suppose that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in W$. Prove that W is all of $\mathbb{R}^2$. (This is easy, as long as you remember that any vector space is closed under vector addition and scalar multiplication. This means that, given any vector is $\mathbb{R}^2$, you just need to be able to write that vector as a linear combination of the two given vectors.)

A vector space is always closed under scalar addition and vector multiplication. That means that if $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are in W, then $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ will be in W for all numbers $x, y \in \mathbb{R}$. So, the question is, can we get every vector in $\mathbb{R}^2$ from linear combinations of this form? Consider a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in R^2. the question is, can we find values for x and y for which

$$x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

or, in terms of equations,

$$\begin{aligned} x - 2y &= a \\ y &= b \end{aligned}$$

But this system clearly has solution $y = b$, $x = a + 2b$.

Problem 6. Remember that to show that a set is **not** a vector space, you only need to find one vector space property that fails, out of the ten properties that vector spaces must satisfy.

(a) Let S be the subset of $\mathbb{R}^2$ defined as $S = \{(x, y) \mid x + y = 1\}$. Show that S, using the addition and scalar multiplication operations from $\mathbb{R}^2$, is **not** a vector space.

(b) Let T be the subset of $\mathbb{R}^2$ defined as $T = \{(x, y) \mid x + y \geq 0\}$. Show that T, using the addition and scalar multiplication operations from $\mathbb{R}^2$, is **not** a vector space.

(a) S does not contain the zero vector, since $0 + 0 \neq 1$, so S is not a vector space. Alternatively, note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S$ but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is not in S, so S is not closed under vector addition and therefore is not a vector space.

(b) Note that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in T$, but $(-1) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, which is not in T, so T is not closed under scalar multiplication and therefore is not a vector space.