Problem 1. Decide whether each set of vectors is a basis for $\mathbb{R}^3$. Give a reason for your answer. In some cases, the reason can be very short. In other cases, a calculation is required.

(a) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right\}$

Answer:

(a) This is not a basis because any basis of $\mathbb{R}^3$ has exactly three elements. (Two vectors cannot span $\mathbb{R}^3$.)

(b) This is not a basis because any basis of $\mathbb{R}^3$ has exactly three elements. (Three vectors in $\mathbb{R}^3$ cannot be linearly independent.)

(c) The second vector is two times the first, so the vectors are not linearly independent and so cannot be a basis.

(d) One way to test whether three vectors in $\mathbb{R}^3$ form a basis is to make the matrix whose columns are the three vectors, and test whether the matrix is non-singular.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 0 & 3 & 5 \end{pmatrix} \xrightarrow{-2\rho_1 + \rho_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 3 & 5 \end{pmatrix}$$

$$\xrightarrow{\rho_2 + \rho_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 3 \end{pmatrix}$$

Since the echelon form of the matrix has no row of zeros, the matrix is non-singular and the vectors do form a basis.

Problem 2. Suppose that $(V, +, \cdot)$ is a vector space and $\mathcal{U}$ is a basis of V, where $\mathcal{U} = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$. Let $\mathcal{D} = \langle \vec{\beta}_n, \vec{\beta}_{n-1}, \ldots, \vec{\beta}_1 \rangle$. (Then $\mathcal{D}$ is also a basis of V. It is a different basis, since these bases are ordered.) For a vector $\vec{v} \in V$, how does $\text{Rep}_{\mathcal{U}}(\vec{v})$ compare to $\text{Rep}_{\mathcal{D}}(\vec{v})$? Justify your answer.

Answer:

If $\text{Rep}_{\mathcal{U}}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, then $\text{Rep}_{\mathcal{D}}(\vec{v}) = \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_1 \end{pmatrix}$. That is, the representation is just reversed top to bottom. Saying that $\text{Rep}_{\mathcal{U}}(\vec{v})$ is as given above just means that

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_{n-1}\vec{\beta}_{n-1} + c_n\vec{\beta}_n = \vec{v}$$

but since vector addition is commutative, it is equally true that

$$c_n\vec{\beta}_n + c_{n-1}\vec{\beta}_{n-1} + \cdots + c_2\vec{\beta}_2 + c_1\vec{\beta}_1 = \vec{v}$$

since that just changes the order of the terms in the sum. This second equation says that $\text{Rep}_{\mathcal{D}}(\vec{v})$ is as given above. (The order of vectors matters in a basis.)

Problem 3. The sequence $\mathcal{B} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ is a basis for $\mathbb{R}^3$. Let $\vec{v} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$.

Find $\text{Rep}_{\mathcal{B}}(\vec{v})$. (You do **not** need to prove that $\mathcal{B}$ is a basis.)

Answer:

We have to find a, b, c such that $a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{v} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$, which equivalent to the system of equations

$$\begin{aligned} a + b + c &= 5 \\ b + c &= 7 \\ c &= 3 \end{aligned}$$

This equation is already in echelon form and is easy to solve:

$$c = 3 \qquad \begin{aligned} b &= 7 - c \\ &= 7 - 3 \\ &= 4 \end{aligned} \qquad \begin{aligned} a &= 5 - b - c \\ &= 5 - 4 - 3 \\ &= -2 \end{aligned}$$

So, $\text{Rep}_{\mathcal{B}}(\vec{v}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

Problem 4. Using the basis, $\mathcal{B}$, from problem 3, suppose that $\text{Rep}_{\mathcal{B}}(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find $\vec{v}$.

Answer:

The vector $\vec{v}$ is equal to a linear combination of the basis vectors, in which the coefficients are the coordinates from the representation vector. That is,

$$\vec{v} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$$

Problem 5. Let $\mathscr{P}_3$ be the vector space of polynomials of degree less than or equal to 3. Let $\mathcal{B}$ be the basis of $\mathscr{P}_3$ given by

$$\mathcal{B} = \langle 1 + x, x - x^2, 1 + x^3, 2x - x^2 + x^3 \rangle$$

Find $\text{Rep}_{\mathcal{B}}(3 - 4x + 4x^2 + x^3)$. (You do **not** have to prove that $\mathcal{B}$ is a basis.)

Answer:

We must find a, b, c, d such that

$$a(1 + x) + b(x - x^2) + c(1 + x^3) + d(2x - x^2 + x^3) = 3 - 4x + 4x^2 + x^3$$

This can be written as

$$(a + c) + (a + b + 2d)x + (-b - d)x^2 + (c + d)x^3 = 3 - 4x + 4x^2 + x^3$$

This give the system of equations

$$\begin{aligned} a + c &= 3 \\ a + b + 2d &= -4 \\ -b - d &= 4 \\ c + d &= 1 \end{aligned}$$

We can solve this using an augmented matrix to represent the system

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 2 & -4 \\ 0 & -1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 & -7 \\ 0 & -1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 & -7 \\ 0 & 0 & -1 & 1 & -3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\rho_3+\rho_4} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 & -7 \\ 0 & 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 2 & -2 \end{array}\right)$$

So we get $d = -1$, $c = 3 + d = 2$, $b = -7 + c - 2d = -3$, and $a = 3 - c = 1$. So, $\text{Rep}_{\mathcal{B}} = \begin{pmatrix} 1 \\ -3 \\ 2 \\ -1 \end{pmatrix}$.