*This homework is due by **noon** on Tuesday, October 13,*

Problem 1. Find the rank of each matrix:

(a) $\begin{pmatrix} 1 & 3 & -2 & 4 \\ 2 & 1 & 1 & 3 \\ -1 & 2 & -3 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 3 & 5 & 2 \\ -1 & 2 & 1 & -2 & 3 \\ 2 & 4 & 0 & 1 & 1 \\ 3 & 4 & 3 & 6 & 3 \\ 1 & 6 & 1 & -1 & 4 \end{pmatrix}$

Answer:

If we put the matrix into echelon form, the rank of the original matrix is just the number of non-zero rows in the echelon form matrix.

(a)

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 2 & 1 & 1 & 3 \\ -1 & 2 & -3 & 1 \end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -5 & 5 & -5 \\ 0 & 5 & -5 & 5 \end{pmatrix}$$

$$\xrightarrow{\rho_2+\rho_3} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are two non-zero rows, the rank is 2.

(b)

$$\begin{pmatrix} 1 & 0 & 3 & 5 & 2 \\ -1 & 2 & 1 & -2 & 3 \\ 2 & 4 & 0 & 1 & 1 \\ 3 & 4 & 3 & 6 & 3 \\ 1 & 6 & 1 & -1 & 4 \end{pmatrix} \xrightarrow{\substack{\rho_1+\rho_2 \\ -2\rho_1+\rho_3 \\ -3\rho_1+\rho_4 \\ -\rho_1+\rho_5}} \begin{pmatrix} 1 & 0 & 3 & 5 & 2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 4 & -6 & -9 & -3 \\ 0 & 4 & -6 & -9 & -3 \\ 0 & 6 & -2 & -6 & 2 \end{pmatrix}$$

$$\xrightarrow{\substack{-2\rho_2+\rho_3 \\ -2\rho_2+\rho_4 \\ -3\rho_2+\rho_5}} \begin{pmatrix} 1 & 0 & 3 & 5 & 2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 0 & -14 & -15 & -13 \\ 0 & 0 & -14 & -15 & -13 \\ 0 & 0 & -14 & -15 & -13 \end{pmatrix}$$

$$\xrightarrow{\substack{-\rho_3+\rho_4 \\ =\rho_3+\rho_5}} \begin{pmatrix} 1 & 0 & 3 & 5 & 2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 0 & -14 & -15 & -13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are three non-zero rows, the rank is 3.

Problem 2. Suppose that $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a homomorphism that satisfies

$$h\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix}, \qquad h\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}3\\-1\\0\end{pmatrix}, \quad \text{and} \quad h\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

(a) Find $h\begin{pmatrix}-2\\3\\1\end{pmatrix}$. (Remember that h is a homomorphism.)

(b) For any vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$, find $h\begin{pmatrix}a\\b\\c\end{pmatrix}$, writing the answer in terms of a, b, and c.

(c) Find a specific vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ such that $h\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$.

Answer:

(a) $h\begin{pmatrix}-2\\3\\1\end{pmatrix} = -2h(\vec{e}_1) + 3h(\vec{e}_2) + h(\vec{e}_3) = -2\begin{pmatrix}0\\2\\1\end{pmatrix} + 3\begin{pmatrix}3\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}10\\-5\\1\end{pmatrix}$

(b) $h\begin{pmatrix}a\\b\\c\end{pmatrix} = ah(\vec{e}_1) + bh(\vec{e}_2) + ch(\vec{e}_3) = a\begin{pmatrix}0\\2\\1\end{pmatrix} + b\begin{pmatrix}3\\-1\\0\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3b+c\\2a-b+2c\\a+3c\end{pmatrix}$

(c) We just need to solve $\begin{pmatrix}3b+c\\2a-b+2c\\a+3c\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$. This is a system of equations that can be solved using row reduction.

$$\left(\begin{array}{ccc|c}0&3&1&-1\\2&-1&2&0\\1&0&3&2\end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{ccc|c}1&0&3&2\\2&-1&2&0\\0&3&1&-1\end{array}\right)$$

$$\xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c}1&0&3&2\\0&-1&-4&-4\\0&3&1&-1\end{array}\right)$$

$$\xrightarrow{3\rho_2+\rho_3} \left(\begin{array}{ccc|c}1&0&3&2\\0&-1&-4&-4\\0&0&-11&-13\end{array}\right)$$

$$\xrightarrow[-\frac{1}{11}\rho_3]{-\rho_2} \left(\begin{array}{ccc|c}1&0&3&2\\0&1&4&4\\0&0&1&\frac{13}{11}\end{array}\right)$$

$$\xrightarrow[]{\substack{-3\rho_3+\rho_1 \\ -4\rho_3+\rho_2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{17}{11} \\ 0 & 1 & 0 & -\frac{8}{11} \\ 0 & 0 & 1 & \frac{13}{11} \end{array}\right)$$

So the solution is $a = -\frac{17}{11}, b = -\frac{8}{11}, c = \frac{13}{11}$, or $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -17 \\ -8 \\ 13 \end{pmatrix}$

Problem 3. In class, we showed that the function from $\mathscr{P}_3$ to $\mathscr{P}_3$ that maps the polynomial $p(x)$ to the polynomial $p(x-1)$ is an automorphism of $\mathscr{P}_3$. Define the homomorphism $h\colon \mathscr{P}_2 \to \mathscr{P}_2$ by $h(p(x)) = p(2x+5)$. (You do not have to show that this function is a homomorphism. Note that it is defined on $\mathscr{P}_2$, not $\mathscr{P}_3$.)

(a) Show that h is bijective by finding an inverse function.

(b) Write out $h(a+bx+cx^2)$ as a polynomial in standard form $(d+ex+fx^2)$, where d, e, f are expressed in terms of a, b, c).

Answer:

(a) $h^{-1}(q(x)) = q(\frac{1}{2}(x-5))$, because $h(h^{-1}(q(x)) = h(q(\frac{1}{2}(x-5))) = q(2(\frac{1}{2}(x-5)+5) = q(x)$ and $h^{-1}(h(p(x)) = h^{-1}(p(2x+5)) = p(\frac{1}{2}((2x+5)-5)) = p(x)$.

(b) $h(a+bx+cx^2) = a + b(2x+5) + c(2x+5)^2) = a + b(2x+5) + c(4x^2+20x+25) = (a+5b+25c) + (2b+20c)x + 4cx^2$.

Problem 4. Define $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ by $f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}$. Show by direct calculation that f is a homomorphism, and show that it is in fact an automorphism by finding its inverse.

Answer:

(1) Show that $h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$:

$$h\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}\right) = h\begin{pmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \\ d_1+d_2 \end{pmatrix} = \begin{pmatrix} d_1+d_2 \\ c_1+c_2 \\ b_1+b_2 \\ a_1+a_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ c_1 \\ b_1 \\ a_1 \end{pmatrix} + \begin{pmatrix} d_2 \\ c_2 \\ b_2 \\ a_2 \end{pmatrix} = h\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + h\begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}$$

(2) Show that $h(r \cdot \vec{v}) = r \cdot h(\vec{v})$:

$$h\left(r \cdot \begin{pmatrix} 1 \\ b \\ c \\ d \end{pmatrix}\right) = h\begin{pmatrix} ra \\ rb \\ rc \\ rd \end{pmatrix} = \begin{pmatrix} rd \\ rc \\ rb \\ ra \end{pmatrix} = r \cdot \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} = r \cdot h\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

(3) Show that h is an automorphism. In fact, $h^{-1} = h$ because $h\left(h\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\right) = h\begin{pmatrix}d\\c\\b\\a\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$.

Since h has an inverse, it is bijective, and hence is an automorphism.

Problem 5. Suppose that V, W, and X are vector spaces and that $f\colon V \to W$ and $g\colon W \to X$ are homomoprhisms. Recall that the compostion, $g \circ f$, of g and f is defined to be the function from V to X given by $g \circ f(\vec{v} = g(f(\vec{v}))$ for $\vec{v} \in V$. Show that $g \circ f$ is a homomorphism. (This is easy! Just check the two conditions for a function to be a homomorphism.)

Answer:

(1) Let $\vec{v}_1, \vec{v}_2 \in V$. Show that $g \circ f(\vec{v}_1 + \vec{v}_2) = g \circ f(\vec{v}_1) + g \circ f(\vec{v}_2)$:

$$g \circ f(\vec{v}_1 + \vec{v}_2) = g(f(\vec{v}_1 + \vec{v}_2)) = g(f(\vec{v}_1) + f(\vec{v}_2)) = g(f(\vec{v}_1)) + g(f(\vec{v}_2)) = g \circ f(\vec{v}_1) + g \circ f(\vec{v}_2)$$

(2) Let $\vec{v} \in V$ and $r \in \mathbb{R}$. Show that $g \circ f(r \cdot \vec{v}) = r \cdot (g \circ f(\vec{v}))$:

$$g \circ f(r \cdot \vec{v}) = g(f(r \cdot \vec{v})) = g(r \cdot f(\vec{v})) = r \cdot g(f(\vec{v})) = r \cdot (g \circ f(\vec{v}))$$