This homework is due by 11:59 PM on Thursday, October 28

Problem 1. Let A be the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Put the matrix A into reduced echelon form. This can be done with four row operations. Now, based on your row reduction, write the matrix A as a product of 3×3 matrices, where each matrix in the product is an elementary matrix.

Answer:

First, we put the matrix into reduced echelon form:

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \xrightarrow{-2\rho_1 + \rho_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & 0 \end{pmatrix} \xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{5}\rho_2} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-3\rho_2 + \rho_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can convert the steps of the row reduction into matrix multiplication, writing the identity as A multiplied on the left by one elementary matrix corresponding to each row operation:

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By multiplying this equation on the left by the inverse of each of the four elementary matrices, we get A wirtten as a product of elementary matrices:

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2. The $n \times n$ identity matrix, I_n, has the property that it is its own inverse. That is, the product $I_n I_n$ is equal to I_n. There are other $n \times n$ matrices that have the same property; that is, $AA = I_n$.

(a) Describe all **diagonal** $n \times n$ matrices D that have the property $DD = I_n$.

(b) Let S be the 2×2 matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the matrix product SS to see that S is its own inverse.

(c) The matrix S from the previous part is a **permutation matrix**; multiplying a $2 \times n$ matrix on the left by S will swap the two rows of that matrix, so SS is the matrix that you get by swapping the rows of S, producing the identity matrix. Find two different 3×3 permutation matrices A and B that are their own inverses. That is, $AA = I_3$ and $BB = I_3$.

(d) Find a 3×3 permutation matrix A that has the property $AAA = I_3$.

Answer:

(a) Suppose that D is an $n \times n$ diagonal matrix and the i^{th} diagonal entry is d_i, for $i = 1, 2, \ldots, n$. Then the i^{th} diagonal entry in the matrix DD is d_i^2. We want to have $DD = I_n$, which means that that the i^{th} diagonal entry must be 1. That is, we want $d_i^2 = 1$. The only solutions of this equation are $d_i = 1$ and $d_i = -1$. So, a diagonal matrix D satisfies $DD = I_n$ if and only if every diagonal entry in D is either 1 or -1. (There are 2^n such matrices.)

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) Any elementary matrix that swaps two rows is a permuation matrix and is its own inverse. So, for example, using the notation introduced in class, we can use

$$R_{\rho_1 \leftrightarrow \rho_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R_{\rho_1 \leftrightarrow \rho_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(d) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 3. Let $d\colon \mathscr{P}_4 \to \mathscr{P}_3$ be the derivative, $d(p(x)) = p'(x)$. Find the matrix $\text{Rep}_{B,D}(d)$ where B and D are the usual bases for $\mathscr{P}_4$ and $\mathscr{P}_3$, $B = \langle 1, x, x^2, x^3 \rangle$ and $D = \langle 1, x, x^2 \rangle$.

Answer:

To get column 1 of the matrix: $d(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$
To get column 2 of the matrix: $d(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$
To get column 3 of the matrix: $d(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$
To get column 4 of the matrix: $d(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$

So, $\text{Rep}_{B,D}(d) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Problem 4. Let h be the homomorphism $h\colon \mathscr{P}_2 \to \mathscr{P}_2$ given by

$$h(a + bx + cx^2) = (a + b) + (b + c)x + (c + a)x^2$$

Let B be the basis of $\mathscr{P}_2$ given by $B = \langle 1, 1 + x, 1 + x + x^2 \rangle$. Find the matrix $\text{Rep}_{B,B}(h)$.

Answer:

To get column 1 of the matrix: $h(1) = 1 + x^2 = 1 \cdot (1) + (-1) \cdot (1 + x) + 1 \cdot (1 + x + x^2)$
To get column 2 of the matrix: $h(1 + x) = 2 + x + x^2 = 1 \cdot (1) + 0 \cdot (1 + x) + 1 \cdot (1 + x + x^2)$
To get column 3 of the matrix: $h(1+x+x^2) = 2+2x+2x^2 = 0 \cdot (1) + 0 \cdot (1+x) + 2 \cdot (1+x+x^2)$

So, $\text{Rep}_{B,B}(h) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

Problem 5. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the homomorphism given by $f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+b \\ 2b-c \end{pmatrix}$. Find the matrix $\text{Rep}_{B,D}$ where the bases B and D of $\mathbb{R}^2$ and $\mathbb{R}^2$ are given by

$$B = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \text{and} \qquad D = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle$$

Answer:

To get column 1 of the matrix: $f\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 11 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

To get column 2 of the matrix: $f\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

To get column 3 of the matrix: $f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

So, $\text{Rep}_{B,D}(f) = \begin{pmatrix} 4 & 1 & 0 \\ 11 & 2 & 1 \end{pmatrix}$

Problem 6. Let V be a vector space with basis $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$. Let $g\colon V \to V$ is a homomorphism. Suppose that there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $g(\vec{\beta}_1) = \lambda_1 \cdot \vec{\beta}_1$, $g(\vec{\beta}_2) = \lambda_2 \cdot \vec{\beta}_2$, $\ldots$, $g(\vec{\beta}_n) = \lambda_n \cdot \vec{\beta}_n$. What is $\text{Rep}_{B,B}(g)$?

(Preview: If $h\colon V \to V$ is a homomorphism and $h(\vec{v}) = \lambda \cdot \vec{v}$ for some $\lambda \in \mathbb{R}$ and $\vec{v} \in V$, then λ is called an **eigenvalue** for h, and $\vec{v}$ is called an **eigenvector** for h with eigenvalue λ. The homomorphism g in this problem admits a **basis of eigenvectors**, but this is not the usual case.)

Answer:

If we write $g(\vec{\beta}_i$ in terms of the basis B, we just have $g(\vec{\beta}_i) = \lambda_i \cdot \vec{\beta}_i$. That is, the coefficient of $\vec{\beta}_i$ is λ_i, and the coefficients of all the other basis vectors are zero. So the i^{th} column of $\text{Rep}_{B,B}(g)$ has λ_i in row i and 0 in all the other rows. This makes $\text{Rep}_{B,B}(g)$ a diagonal matrix, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$.

$$\text{Rep}_{B,B}(g) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$