This homework is due by 11:59 PM on Tuesday, November 10

Problem 1. Remember that the change of basis matrix for bases B and D of the same vector space V is defined to be $Rep_{B,D}(id)$, where $id\colon V \to V$ is the identity map.

Suppose that B and D are bases for $\mathscr{P}_2$, where $D = \langle x + x^2, 1 - 2x + x^2, 2 - x\rangle$. Find the basis B if the change of basis matrix is $\text{Rep}_{B,D}(id) = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 1 & 4 \end{pmatrix}$. (This problem is very easy!)

Answer:

Write $B = \langle p_1(x), p_2(x), p_3(x)\rangle$. When we write $p_i(x)$ as a linear combination of elements of the basis D, the coefficients in the linear combination come form the i^{th} column of the matrix $\text{Rep}_{B,D}(id)$, so we must have

$$\begin{aligned} p_1(x) &= (-1)\cdot(x + x^2) + 2\cdot(1 - 2x + x^2) + 0\cdot(2 - x) \\ &= 2 - 5x + x^2 \end{aligned}$$

$$\begin{aligned} p_2(x) &= 2\cdot(x + x^2) + 3\cdot(1 - 2x + x^2) + 1\cdot(2 - x) \\ &= 5 - 5x + 5x^2 \end{aligned}$$

$$\begin{aligned} p_3(x) &= 0\cdot(x + x^2) + (-1)\cdot(1 - 2x + x^2) + 4\cdot(2 - x) \\ &= 7 - 2x - x^2 \end{aligned}$$

So, $B = \langle 2 - 5x + x^2, 5 - 5x + 5x^2, 7 - 2x - x^2\rangle$.

Problem 2. Consider the bases B and D for $\mathbb{R}^3$, as given here:

$$B = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix} \right\rangle \qquad D = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

(a) Find the change of basis matrix $\text{Rep}_{B,D}(id)$.

(b) Check that $\text{Rep}_B \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. (You're not asked to find the representation, just check it.)

(c) The change of basis matrix must satisfy $\text{Rep}_{B,D}(id)\cdot\text{Rep}_B(\vec{v}) = \text{Rep}_D(\vec{v})$. Verify that in fact

$$\text{Rep}_{B,D}(id)\cdot\text{Rep}_B \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \text{Rep}_D \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

Answer:

(a) The columns of $\text{Rep}_{B,D}(id)$ come from the coefficients when we write the elements of B as linear combinations of the elements of D. We can do that by inspection:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} + (-2) \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} + (-4) \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\-3\\2\end{pmatrix} = 0 \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} + (-2) \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

$$\begin{pmatrix}-2\\4\\-3\end{pmatrix} = (-2) \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} + (-3) \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

so the representation matrix is $\text{Rep}_{B,D}(id) = \begin{pmatrix}1 & 0 & -2\\-2 & -2 & 1\\-4 & 1 & -3\end{pmatrix}$.

(b) The entries in the vector $\text{Rep}_B\begin{pmatrix}-1\\0\\4\end{pmatrix}$ are obtained by writing the vector as a linear combination of elements of B. The coefficients in the linear combination are the entries in the representation vector. So we just need to check that

$$\begin{pmatrix}-1\\0\\4\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} + 2 \cdot \begin{pmatrix}0\\-3\\2\end{pmatrix} + 1 \cdot \begin{pmatrix}-2\\4\\-3\end{pmatrix}$$

which is true.

(c) $\text{Rep}_{B,D}(id) \cdot \text{Rep}_B\begin{pmatrix}-1\\0\\4\end{pmatrix} = \begin{pmatrix}1 & 0 & -2\\-2 & -2 & 1\\-4 & 1 & -3\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-5\\-5\end{pmatrix}$, so we just need to check that $\begin{pmatrix}-1\\-5\\-5\end{pmatrix} = \text{Rep}_D\begin{pmatrix}-1\\0\\4\end{pmatrix}$. But that is true using the same reasoning as in part (b) because

$$\begin{pmatrix}-1\\0\\4\end{pmatrix} = (-1) \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} + (-5) \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} + (-5) \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix}$$

Problem 3. Find an affine transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix} \qquad f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$$

Hint: It's easy to find the translation part of the affine map! Recall that f can be represented in the form $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ax+by+e\\cx+dy+f\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}e\\f\end{pmatrix}$.

Answer:

Using the formula for f, we have $\begin{pmatrix}-2\\3\end{pmatrix} = f\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}e\\f\end{pmatrix}$, so we must have $e = -2$ and $f = 3$.

Then, rewriting $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-2\\3\end{pmatrix}$ as $\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = f\begin{pmatrix}x\\y\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix}$, we get that

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = f\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$$

which means $a = 3$ and $c = -2$. Similarly,

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = f\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$$

which means $b = 4$ and $d = 2$. So the affine transformation is

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3x+4y-2\\-2x+2y+3\end{pmatrix}$$

Problem 4. The cross product of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ is a vector, $\vec{v} \times \vec{w}$, that is orthogonal to both $\vec{v}$ and $\vec{w}$. The cross product can be computed as the "formal determinant"

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\a_1 & a_2 & a_3\\b_1 & b_2 & b_3\end{vmatrix}$$

Find $\begin{pmatrix}3\\-1\\2\end{pmatrix} \times \begin{pmatrix}1\\4\\-2\end{pmatrix}$ by writing out the formal determinant $\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\3 & -1 & 2\\1 & 4 & -2\end{vmatrix}$, using the formula for the determinant of a 3×3 matrix, and show that the result is, in fact, orthogonal to both vectors.

Answer:

$$\begin{aligned}\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\3 & -1 & 2\\1 & 4 & -2\end{vmatrix} &= \vec{e}_1 \cdot (-1) \cdot (-2) + \vec{e}_2 \cdot 2 \cdot 1 + \vec{e}_3 \cdot 3 \cdot 4 - \vec{e}_1 \cdot 2 \cdot 4 - \vec{e}_2 \cdot 3 \cdot (-2) - \vec{e}_3 \cdot 1 \cdot (-1)\\ &= 2\vec{e}_1 + 2\vec{e}_2 + 12\vec{e}_3 - 8\vec{e}_1 + 6\vec{e}_2 + 1\vec{e}_3\\ &= -6e_1 + 8\vec{e}_2 + 13\vec{e}_3\end{aligned}$$

$$= \begin{pmatrix} -6 \\ 8 \\ 13 \end{pmatrix}$$

and we can check that the matrices are orthogonal by computing dot products:

$$\begin{pmatrix} -6 \\ 8 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = -6 \cdot 3 - 8 \cdot 1 + 2 \cdot 13 = 0 \qquad \begin{pmatrix} -6 \\ 8 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = -6 \cdot 1 + 8 \cdot 4 - 2 \cdot 13 = 0$$

Problem 5. We looked at a formula for computing the determinant of a 3×3 matrix. That formula can be derived using Laplace's expansion for the determinant. Apply Laplace's expansion to the general 3×3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

to derive the formula for the determinant of a 3×3 matrix. (You only need to apply Laplace's expansion for the first step of the computation, not for the resulting 2×2 matrices.)

Answer:

The calculation using Laplace's expansion for the 3×3 matrix produces the same formula that we already saw for the determinant:

$$\begin{aligned}
\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\
&= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \\
&= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1
\end{aligned}$$

Problem 6. Compute each of the following determinants. Some of them are very easy, using properties of the determinant, and you should check for that before doing a complex computation. For any problem where you use a property of the determinant to find the answer, state the property that you use.

(a) $\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 5 & 7 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix}$ **(c)** $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

(d) $\begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & -3 \\ 4 & 5 & -1 \end{vmatrix}$ **(e)** $\begin{vmatrix} 1 & 2 & 4 & -1 \\ 3 & 5 & -3 & 7 \\ 1 & 2 & 4 & -1 \\ 6 & 2 & 3 & 7 \end{vmatrix}$ **(f)** $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 5 & 6 & 1 \\ -1 & 1 & 1 & 3 \end{vmatrix}$

Answer:

(a) Using the formula for a 2×2 matrix, $\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix} = 3 \cdot (-1) - 2 \cdot 5 = -13$

(b) Using the fact that the determinant of an echelon form matrix is the product of the diagonal entries, $\begin{vmatrix} 5 & 7 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 5 \cdot 2 \cdot 3 = 30$

(c) Using row reduction to put the matrix into echelon form,

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -5 & 5 \\ 0 & -7 & 10 \end{vmatrix} = -5 \begin{vmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & -7 & 10 \end{vmatrix} = -5 \begin{vmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = -5 \cdot 3 = -15$$

or using the formula for the determinant of a 3×3 matrix, we get

$$1 \cdot 1 \cdot 4 + 3 \cdot 1 \cdot 3 + (-2) \cdot 2 \cdot 2 - (-2) \cdot 1 \cdot 3 - 3 \cdot 2 \cdot 4 - 1 \cdot 1 \cdot 2 = 4 + 9 - 8 + 6 - 24 - 2 = -15$$

(d) Swapping the first and last rows changes the sign of the determinant. Then we can apply the fact that the determinant of an echelon form matrix is the product of the diagonal entries:

$$\begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & -3 \\ 4 & 5 & -1 \end{vmatrix} = - \begin{vmatrix} 4 & 5 & -1 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{vmatrix} = -3 \cdot 2 \cdot 4 = -24$$

(e) The determinant is 0 because there are two identical rows (the first and third).

(f) Using row reduction to put the matrix into echelon form,

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 2 & 5 & 6 & 1 \\ -1 & 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 3 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 10 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 10 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & -19 \end{vmatrix}$$

$$= 1 \cdot (-1) \cdot 2 \cdot (-19) = 38$$