Problem 1. Some of the following matrix products are not defined. For each product, you should either compute the product, if it is defined, or state why it is not defined.

a) \[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
3 & -1 \\
-2 & 2 \\
1 & 0
\end{pmatrix}
\]
b) \[
\begin{pmatrix}
0 & 3 & 5 \\
2 & 1 & 7 \\
1 & 3 & 0
\end{pmatrix}
\begin{pmatrix}
5 & 2 \\
3 & 1
\end{pmatrix}
\]
c) \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
d) \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
e) \[
\begin{pmatrix}
3 \\
5
\end{pmatrix}
\begin{pmatrix}
3 & 2 \\
1 & 7
\end{pmatrix}
\]

Problem 2. Let \( A = \begin{pmatrix}1 & 2 \\ 3 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix}1 & 2 & 3 & 4 \end{pmatrix} \). Note that \( A \in M_{4 \times 1} \) and \( B \in M_{1 \times 4} \). Compute the matrix products \( AB \) and \( BA \).

Problem 3. Find the inverse of each of the following matrices, or show that the matrix has no inverse. For part (c), you should find the inverse by putting the augmented matrix
\[
\begin{pmatrix}
-1 & 3 & 0 & 1 & 0 & 0 \\
2 & -1 & 5 & 0 & 1 & 0 \\
1 & 2 & -5 & 0 & 0 & 1
\end{pmatrix}
\]
into reduced echelon form.

a) \[
\begin{pmatrix}
3 & -2 \\
5 & 4
\end{pmatrix}
\]
b) \[
\begin{pmatrix}
6 & -4 \\
-3 & 2
\end{pmatrix}
\]
c) \[
\begin{pmatrix}
-1 & 3 & 0 \\
2 & -1 & 5 \\
1 & 2 & -5
\end{pmatrix}
\]

Problem 4. We can define the derivative for all polynomials as a homomorphism \( D: \mathcal{P} \rightarrow \mathcal{P} \). The null space \( \mathcal{N}(D) \) is the set of all constant polynomials. We can form the composition \( D \circ D \), which also maps \( \mathcal{P} \) to \( \mathcal{P} \). For a polynomial \( q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \), what is \( D \circ D(q(x)) \)? What mathematical operation does \( D \circ D \) compute? What is the null space of \( D \circ D \)?

Problem 5. Suppose that \( V \) and \( W \) are vector spaces and that \( f: V \rightarrow W \) is a homomorphism. Suppose that \( S \subseteq V \) and that \( S \) spans \( V \). Show that the set \( f(S) \) spans the range space \( \mathcal{R}(f) \) of \( f \). (Note: \( \mathcal{R}(f) = \{ f(\vec{v}) \mid \vec{v} \in V \} \), and \( f(S) = \{ f(\vec{v}) \mid \vec{v} \in S \} \).)

Problem 6. Suppose that \( V \) and \( W \) are vector spaces and that \( f: V \rightarrow W \) is a homomorphism. Assume that \( h \) is one-to-one (which implies that the null space \( \mathcal{N}(h) \) contains only \( \vec{0}_V \)). And suppose that \( \langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \rangle \) is a linearly independent sequence of vectors in \( V \). Show that the sequence \( \langle f(\vec{v}_1), f(\vec{v}_2), \ldots, f(\vec{v}_k) \rangle \) is linearly independent (in \( W \)).