This homework is due by 11:59 PM on Thursday, October 29

Problem 1. Let A be the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Put the matrix A into reduced echelon form. This can be done with four row operations. Now, based on your row reduction, write the matrix A as a product of 3×3 matrices, where each matrix in the product is an elementary matrix.

Problem 2. The $n \times n$ identity matrix, I_n, has the property that it is its own inverse. That is, the product $I_n I_n$ is equal to I_n. There are other $n \times n$ matrices that have the same property; that is, $AA = I_n$.

(a) Describe all **diagonal** $n \times n$ matrices D that have the property $DD = I_n$.

(b) Let S be the 2×2 matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the matrix product SS to see that S is its own inverse.

(c) The matrix S from the previous part is a **permutation matrix**; multiplying a $2 \times n$ matrix on the left by S will swap the two rows of that matrix, so SS is the matrix that you get by swapping the rows of S, producing the identity matrix. Find two different 3×3 permutation matrices A and B that are their own inverses. That is, $AA = I_3$ and $BB = I_3$.

(d) Find a 3×3 permutation matrix A that has the property $AAA = I_3$.

Problem 3. Let $d \colon \mathscr{P}_4 \to \mathscr{P}_3$ be the derivative, $d(p(x)) = p'(x)$. Find the matrix $\text{Rep}_{B,D}(d)$ where B and D are the usual bases for $\mathscr{P}_4$ and $\mathscr{P}_3$, $B = \langle 1, x, x^2, x^3 \rangle$ and $D = \langle 1, x, x^2 \rangle$.

Problem 4. Let h be the homomorphism $h \colon \mathscr{P}_2 \to \mathscr{P}_2$ given by

$$h(a + bx + cx^2) = (a + b) + (b + c)x + (c + a)x^2$$

Let B be the basis of $\mathscr{P}_2$ given by $B = \langle 1, 1 + x, 1 + x + x^2 \rangle$. Find the matrix $\text{Rep}_{B,B}(h)$.

Problem 5. Let $f \colon \mathbb{R}^3 \to \mathbb{R}^2$ be the homomorphism given by $f\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + b \\ 2b - c \end{pmatrix}$. Find the matrix $\text{Rep}_{B,D}$ where the bases B and D of $\mathbb{R}^2$ and $\mathbb{R}^2$ are given by

$$B = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \text{and} \qquad D = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle$$

Problem 6. Let V be a vector space with basis $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$. Let $g \colon V \to V$ is a homomorphism. Suppose that there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $g(\vec{\beta}_1) = \lambda_1 \cdot \vec{\beta}_1$, $g(\vec{\beta}_2) = \lambda_2 \cdot \vec{\beta}_2$, $\ldots$, $g(\vec{\beta}_n) = \lambda_n \cdot \vec{\beta}_n$. What is $\text{Rep}_{B,B}(g)$?

(Preview: If $h \colon V \to V$ is a homomorphism and $h(\vec{v}) = \lambda \cdot \vec{v}$ for some $\lambda \in \mathbb{R}$ and $\vec{v} \in V$, then λ is called an **eigenvalue** for h, and $\vec{v}$ is called an **eigenvector** for h with eigenvalue λ. The homomorphism g in this problem admits a **basis of eigenvectors**, but this is not the usual case.)