Problem 1. Let $A$ be the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. Put the matrix $A$ into reduced echelon form. This can be done with four row operations. Now, based on your row reduction, write the matrix $A$ as a product of $3 \times 3$ matrices, where each matrix in the product is an elementary matrix.

Problem 2. The $n \times n$ identity matrix, $I_n$, has the property that it is its own inverse. That is, the product $I_n I_n$ is equal to $I_n$. There are other $n \times n$ matrices that have the same property; that is, $AA = I_n$.

(a) Describe all diagonal $n \times n$ matrices $D$ that have the property $DD = I_n$.

(b) Let $S$ be the $2 \times 2$ matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the matrix product $SS$ to see that $S$ is its own inverse.

(c) The matrix $S$ from the previous part is a permutation matrix; multiplying a $2 \times n$ matrix on the left by $S$ will swap the two rows of that matrix, so $SS$ is the matrix that you get by swapping the rows of $S$, producing the identity matrix. Find two different $3 \times 3$ permutation matrices $A$ and $B$ that are their own inverses. That is, $AA = I_3$ and $BB = I_3$.

(d) Find a $3 \times 3$ permutation matrix $A$ that has the property $AAA = I_3$.

Problem 3. Let $d: \mathcal{P}_4 \to \mathcal{P}_3$ be the derivative, $d (p(x)) = p'(x)$. Find the matrix $\text{Rep}_{B, D} (d)$ where $B$ and $D$ are the usual bases for $\mathcal{P}_4$ and $\mathcal{P}_3$, $B = \langle 1, x, x^2, x^3 \rangle$ and $D = \langle 1, x, x^2 \rangle$.

Problem 4. Let $h$ be the homomorphism $h: \mathcal{P}_2 \to \mathcal{P}_2$ given by

$h(a + bx + cx^2) = (a + b) + (b + c)x + (c + a)x^2$

Let $B$ be the basis of $\mathcal{P}_2$ given by $B = \langle 1, 1 + x, 1 + x + x^2 \rangle$. Find the matrix $\text{Rep}_{B, B} (h)$.

Problem 5. Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be the homomorphism given by $f \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + b \\ 2b - c \end{pmatrix}$. Find the matrix $\text{Rep}_{B, D}$ where the bases $B$ and $D$ of $\mathbb{R}^2$ and $\mathbb{R}^2$ are given by

$B = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$  \quad \text{and}  \quad D = \left\langle \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\rangle$

Problem 6. Let $V$ be a vector space with basis $B = \langle \vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_n \rangle$. Let $g: V \to V$ is a homomorphism. Suppose that there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $g(\vec{\beta}_1) = \lambda_1 \cdot \vec{\beta}_1$, $g(\vec{\beta}_2) = \lambda_2 \cdot \vec{\beta}_2$, $\ldots$, $g(\vec{\beta}_n) = \lambda_n \cdot \vec{\beta}_n$. What is $\text{Rep}_{B, B} (g)$?

(Preview: If $h: V \to V$ is a homomorphism and $h(\vec{v}) = \lambda \cdot \vec{v}$ for some $\lambda \in \mathbb{R}$ and $\vec{v} \in V$, then $\lambda$ is called an eigenvalue for $h$, and $\vec{v}$ is called an eigenvector for $h$ with eigenvalue $\lambda$. The homomorphism $g$ in this problem admits a basis of eigenvectors, but this is not the usual case.)