Problem 1 (Textbook problem 1.4.12a). Suppose that λ is the least upper bound of some set S, and that λ is *not* in S. Prove that λ is an accumulation point of S. [Hint: For any $\varepsilon > 0$, there is a point $s \in S$ such that $\lambda - \varepsilon < s < \lambda$. Now use the definition of accumulation point to finish the proof.]

Answer:

Suppose that S is a set with least upper bound λ, and $\lambda \notin S$. To show that λ is an accumulation point of S, we need to show that for every $\varepsilon > 0$, there is some $s \in S$ with $0 < |\lambda - s| < \varepsilon$. Since $\lambda = lub(S)$, we know that there is some $s \in S$ such that $s > \lambda - \varepsilon$, since otherwise, $\lambda - \varepsilon$ would be a smaller upper bound for S. (This result was also previously proved as a theorem.) Since $s \in S$ and $\lambda \notin S$, we know that $s \neq \lambda$. So in fact, $\lambda - \varepsilon < s < \lambda$, which means $|\lambda - s| > 0$ and $|\lambda - s| < \varepsilon$.

Problem 2 (Textbook problems 1.4.9 and 1.4.10). **(a)** Prove lemma 1.4.5: If x is an accumulation point of a set S and if $\varepsilon > 0$, then there is an infinite number of points of S within distance ε of S. [Hint: Suppose that for some $\varepsilon > 0$, there were only a finite number of points, $s_1, s_2, \ldots, s_k$, of S within ε of x, but not equal to x. Let $\varepsilon' = \min(|s_1 - x|, |s_2 - x|, \ldots, |s_k - x|)$. Now, show that no $s \in S$ satisfies $0 < |s - x| < \varepsilon'$.] **(b)** Deduce that if S is a **finite** subset of $\mathbb{R}$, then S has no accumulation points. [This is trivially a corollary of the lemma.]

Answer:

(a) Suppose x is an accumulation point of S and $\varepsilon > 0$. Suppose, for the sake of contradiction, that there are only finitely many points of S within distance ε of x. Let those points be $s_1, s_2, \ldots, s_k$ (where we omit x from the list if it happens to be in S). Let $\varepsilon' = \min(|x - s_1|, |x - s_2|, \ldots, |x - s_k|)$. Note that $\varepsilon' > 0$. Take any point z that satisfies $0 < |x - z| < \varepsilon'$. Since $|x - z| < \varepsilon'$, while for $i = 1, 2, \ldots, k$, $|x - s_i| \geq \varepsilon'$, we see that z cannot be one of the points $s_1, s_2, \ldots, s_k$. This means that z is not in S. So, there are no points of S within ε' of x (except possibly x itself). This means x is not an accumulation point of S, which is a contradiction. So, there must be infinitely many points of S within ε of x.

(b) If a set S has an accumulation point, then by part (a), there must be infinitely many points of S within distance 1 of x (letting $\varepsilon = 1$ in (a)). But that means S is infinite, not finite.

Problem 3. Prove directly from the epsilon-delta definition of limits, that $\lim_{x \to 5} \frac{2x+4}{7} = 2$.

Answer:

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that $0 < |x - 5| < \delta$ implies $\left|\frac{2x+4}{7} - 2\right| < \varepsilon$. Let $\delta = \frac{7\varepsilon}{2}$. Then for any x satisfying $|x - 5| < \delta$, we have $\left|\frac{2x+4}{7} - 2\right| = \left|\frac{(2x+4)-14}{7}\right| = \left|\frac{2x-10}{7}\right| = \frac{2}{7}|x - 5| < \frac{2}{7} \cdot \frac{7\varepsilon}{2} = \varepsilon$.

Problem 4. Show directly, without using the product rule for limits, that $\lim_{x\to 3} x^3 = 27$. (Note that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.)

Answer:

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that $0 < |x-3| < \delta$ implies $|x^3 - 27| < \varepsilon$. Let $\delta = \min\left(1, \frac{\varepsilon}{37}\right)$. Then for any x satisfying $|x-3| < \delta$, we have $|x-3| < 1$, which means $-1 < x-3 < 1$ or $2 < x < 4$. So, in particular, $|x| = x < 4$. We then have

$$\begin{aligned}
|x^3 - 27| &= |(x^2+3x+9)(x-3)| = |x^2+3x+9||x-3| \\
&\le (|x^2| + |3x| + |9|)|x-3| \\
&= (|x|^2 + 3|x| + 9)|x-3| \\
&< (4^2 + 3\cdot 4 + 9)|x-3| \\
&< 37 \cdot \tfrac{\varepsilon}{37} \\
&= \varepsilon
\end{aligned}$$

Problem 5 (Textbook problem 2.2.9)**.** Suppose that $f(x) \le 0$ for all x in some open interval containing a, except possibly at a. Suppose that $\lim_{x\to a} f(x) = L$. Show that $L \le 0$. [Hint: Assume instead that $L > 0$. Let $\varepsilon = L/2$ and derive a contradiction.]

Answer:

Suppose, for the sake of contradiction, that $\lim_{x\to a} = L$ where $L > 0$. Then $\frac{L}{2} > 0$, so we can find $\delta > 0$ such that for all x satisfying $0 < |x-a| < \delta$, $|f(x) - L| < \frac{L}{2}$. That is, $-\frac{L}{2} < f(x) - L < \frac{L}{2}$. Adding $\frac{L}{2}$, $\frac{L}{2} < f(x) < \frac{3L}{2}$. In particular, we see that for any x satisfying $0 < |x-a| < \delta$, $f(x) > \frac{L}{2} > 0$. This contradicts the fact that $f(x) \le 0$ for all x near enough to a.

Problem 6. This problem gives an alternative proof of the product rule.

(a) Suppose $\lim_{x\to a} f(x) = L$. Show directly from the definition of limit (without using the product rule) that $\lim_{x\to a} f(x)^2 = L^2$.

(b) Verify algebraically, by expanding the right-hand side, that $ab = \frac{1}{4}\big((a+b)^2 - (a-b)^2\big)$.

(c) Let's say that the sum, difference, and constant multiple rules for limits have already been proved, in addition to parts (a) and (b) of this problem. Using all that (and **not** the definition of derivative), prove the product rule for limits.

Answer:

(a) Suppose $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. We want to find $\delta > 0$ such that for all x, $0 < |x-a| < \delta$ implies $|f(x)^2 - L^2| < \varepsilon$.

Since $\lim_{x\to a} f(x) = L$, there is a $\delta_1 > 0$ such that for $0 < |x-a| < \delta_1$, $|f(x) - L| < 1$ and therefore $|f(x)| < |L| + 1$. And there is a $\delta_2 > 0$ such that for $0 < |x-a| < \delta_1$, $|f(x) - L| < \frac{\varepsilon}{2|L|+1}$.

Let $\delta = \min(\delta_1, \delta_2)$. Take any x satisfying $0 < |x-a| < \delta$. Then we have both $|f(x)| \le |L| + 1$, and $|f(x) - L| < \frac{\varepsilon}{2|L|+1}$. So

$$\begin{aligned}
|f(x)^2 - L^2| &= |f(x) + L||f(x) - L| \\
&\le (|f(x) + |L|)|f(x) - L| \\
&< ((|L| + 1) + |L|)|f(x) - L| \\
&< (2|L| + 1)\frac{\varepsilon}{2|L|+1} \\
&= \varepsilon
\end{aligned}$$

(b) This is a simple calculation:

$$\begin{aligned}
\tfrac{1}{4}\big((a+b)^2 - (a-b)^2\big) &= \tfrac{1}{4}\big((a^2 + 2ab + b^2) - (a^2 - 2ab + b^2)\big) \\
&= \tfrac{1}{4}(4ab) \\
&= ab
\end{aligned}$$

(c) Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then, by part (a) and the sum rule,

$$\begin{aligned}
\lim_{x\to a} \big(f(x) + g(x)\big)^2 &= \big(\lim_{x\to a}(f(x) + (g(x))\big)^2 \\
&= \big(\lim_{x\to a} f(x) + \lim_{x\to a} g(x)\big)^2 \\
&= (L+M)^2
\end{aligned}$$

Similarly, by part (a) and the difference rule,

$$\lim_{x\to a} \big(f(x) - g(x)\big)^2 = (L-M)^2$$

And then by the constant multiple and difference rules,

$$\begin{aligned}
\lim_{x\to a} f(x)g(x) &= \lim_{x\to a} \tfrac{1}{4}\big((f(x) + g(x))^2 - (f(x) - g(x)^2\big) \\
&= \tfrac{1}{4}\big((L+M)^2 - (L-M)^2\big) \\
&= LM
\end{aligned}$$