Problem 1 (Textbook problem 1.4.12a). Suppose that $\lambda$ is the least upper bound of some set $S$, and that $\lambda$ is not in $S$. Prove that $\lambda$ is an accumulation point of $S$. [Hint: For any $\varepsilon > 0$, there is a point $s \in S$ such that $\lambda - \varepsilon < s < \lambda$. Now use the definition of accumulation point to finish the proof.]

Problem 2 (Textbook problems 1.4.9 and 1.4.10). (a) Prove lemma 1.4.5: If $x$ is an accumulation point of a set $S$ and if $\varepsilon > 0$, then there is an infinite number of points of $S$ within distance $\varepsilon$ of $x$. That is, $(x - \varepsilon, x + \varepsilon) \cap X$ is infinite. [Hint: Given $\varepsilon > 0$, suppose that there is only a finite number of points, $s_1, s_2, \ldots, s_k$, of $S$ within $\varepsilon$ of $x$, but not equal to $x$. Let $\varepsilon' = \min(|s_1 - x|, |s_2 - x|, \ldots, |s_k - x|)$. Now, show that no $s \in S$ satisfies $0 < |s - x| < \varepsilon'$.] (b) Deduce that if $S$ is a finite subset of $\mathbb{R}$, then $S$ has no accumulation points. [This is trivially a corollary of the lemma.]

Problem 3. Prove directly, using the (epsilon-delta) definition of limits, that $\lim_{x \to 5} \frac{2}{x+1} = 2$.

Problem 4. Show directly, without using the product rule for limits, that $\lim_{x \to 3} x^3 = 27$. (Note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.)

Problem 5. Suppose that $\lim_{x \to a} f(x) = L$ and $c \in \mathbb{R}$. Prove directly, using the definition of limit, that $\lim_{x \to a} cf(x) = cL$. [Be careful: $c = 0$ is a special case.]

Problem 6 (Textbook problem 2.2.9). Suppose that $f(x) \leq 0$ for all $x$ in some open interval containing $a$, except possibly at $a$. Suppose that $\lim_{x \to a} f(x) = L$. Show that $L \leq 0$. [Hint: Assume instead that $L > 0$. Let $\varepsilon = L/2$ and derive a contradiction.] (Remark: A similar proof shows that if $f(x) \geq 0$ for all $x$ near $a$, then $\lim_{x \to a} f(x) \geq 0$, if the limit exists.)

Problem 7. This problem gives an alternative proof of the product rule. (a) Suppose $\lim_{x \to a} f(x) = L$. Show directly from the definition of limit (without using the product rule) that $\lim_{x \to a} f(x)^2 = L^2$.

(b) Verify algebraically, by expanding the right-hand side, that $ab = \frac{1}{4}[(a + b)^2 - (a - b)^2]$.

(c) Let’s say that the sum, difference, and constant multiple rules for limits have already been proved, in addition to parts (a) and (b) of this problem. Using all that (and not the definition of derivative), prove the product rule for limits.