Problem 1. Suppose that $f(x)$ is defined and bounded on an open interval containing 0, except possibly at 0 itself. (That is, there is a number B such that $|f(x)| < B$ for all x in that interval, except possibly $x = 0$.) Show that $\lim_{x\to 0} xf(x) = 0$. [Hint: The product rule does not apply here. Use the Squeeze Theorem and the fact that $|x|$ is a continuous function.]

Answer:

On the open interval where f is defined, we have that $|xf(x)| = |x||f(x)| < |x|B$. (Note that B must be greater than zero, or else $|f(x)| < B$ would be impossible.) This inequality is equivalent to $-B|x| < xf(x) < B|x|$. Since $|x|$ is a coninuous function of x and any constant multiple of a continuous function is continuous, we know that the functions $-B|x|$ and $B|x|$ are both continous. So, $\lim_{x\to 0}(-B|x|) = \lim_{x\to 0} B|x| = B|0| = 0$. Applying the Squeeze Theorem to $-B|x| < xf(x) < B|x|$, we see that $\lim_{x\to 0} xf(x) = 0$.

Problem 2. If $f(x)$ is a continuous function, then we know that $|f(x)|$ is also continuous, since it is a composition of continuous functions. Give a counterexample to show that the converse does not hold. That is, find a function $f(x)$ such that $|f(x)|$ is continuous, but $f(x)$ is not continuous.

Answer:

Let $E(x) = D(x) - \frac{1}{2}$, where $D(x)$ is the Dirichlet function. That is,

$$E(x) = \begin{cases} 1/2 & \text{if } x \text{ is rational} \\ -1/2 & \text{if } x \text{ is irrational} \end{cases}$$

$E(x)$ is not continuous anywhere. But $|E(x)|$ is the constant function, $|E(x)| = \frac{1}{2}$, which is continuous everywhere.

For a simpler example, define

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Then f is not continuous at 0, but $|f(x)| = 1$ for all x and so is continuous.

Problem 3 (Textbook problem 2.5.7)**.** Suppose that f is continuous at a and that $f(a) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all x in the interval $(a - \delta, a + \delta)$.

Answer:

Suppose that f is continuous at $x = a$. Let $\varepsilon = f(a)$, which is greater than zero by assumption. From the definition of continuity, we can find a a $\delta > 0$ such that for any x, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon = f(a)$. This inequality is equivalent to $-f(a) < f(x) - f(a) < f(a)$. Adding $f(a)$ to the inequality $-f(a) < f(x) - f(a)$ gives $0 < f(x)$. So for any $x \in (a - \delta, a + \delta)$, we have $f(x) > 0$.

Problem 4 (Textbook problem 2.4.10)**.** Prove: If $\lim_{x \to a^+} f(x) = L$ and if $c(x)$ is a function such that $a < c(x) < x$ for all x in some interval (a, b), then $\lim_{x \to a^+} f(c(x)) = L$. [Hint: This is confusing but actually easy.]

Answer:

Let $\varepsilon > 0$. We must find $\delta > 0$ such that for any x, if $0 < x - a < \delta$, then $|f(c(x)) - L| < \varepsilon$. Since $\lim_{x \to a^+} f(x) = L$, we know that there is a δ such that for any y, if $0 < y - a < \delta$, then $|f(y) - L| < \varepsilon$ (*). We can take $\delta \leq b - a$, so that we know that $a < c(x) < x$ for all x satisfying $a < x < a + \delta$.

Using the same δ, suppose that $0 < x - a < \delta$. That is $a < x < a + \delta$, so we know by our assumption that $a < c(x) < x$. So we get that $a < c(x) < x < a + \delta$, which gives $a < c(x) < a + \delta$ and then $0 < c(x) - a < \delta$. Applying (*) with $y = c(x)$, we get $|f(c(x)) - L| < \varepsilon$, which is what we needed to show.

Problem 5. Let f be a continuous function on the interval $[a, b]$, and suppose that $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$. Show that f is constant on $[a, b]$. [Hint: Use the Intermediate Value Theorem.]

Answer:

Suppose, for the sake of contradiction, that $f(x)$ is not constant. Then there are points x_1 and x_2 in $[a, b]$ such that $f(x_1) \neq f(x_2)$. Without loss of generality, we can take $x_1 < x_2$. Now, f is continuous on the interval $[x_1, x_2]$, and so satisfies the Intermediate Value Theorem there. Since $f(x_1) \neq f(x_2)$, we know by the density of the irrational numbers that there is some irrational number y between $f(x_1)$ and $f(x_2)$. By the IVT, there must exist some $c \in [x_1, x_2]$ such that $f(c) = y$. But this contradicts the assumption that $f(x) \in \mathbb{Q}$ for all $x \in [a, b]$. So, in fact, f must be constant.

Problem 6 (Textbook problem 2.6.7b)**.** Show that $p(x) = x^4 - x^3 + x^2 + x - 1$ has at least two roots in the interval $[-1, 1]$.

Answer:

Since p is a polynomial, it is continuous everywhere, and the Intermediate Value Theorem will apply to p on any closed, bounded interval. Note that $p(-1) = 1$, $p(0) = -1$, and $p(1) = 1$. Since $p(-1) > 0 > p(0)$, then by the IVT applied to p on the interval $[-1, 0]$, $p(a) = 0$ for some $a \in (-1, 0)$. Since $p(0) < 0 < p(1)$, then by the IVT applied to p on the interval $[0, 1]$, $p(b) = 0$ for some $b \in (-1, 0)$. So p has at least the roots a and b in the interval $[-1, 1]$.

Problem 7. Show that any linear function $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$.

Answer:

Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$ it follows that $|(mx + b) - (my + b)| < \varepsilon$. In the case $m \neq 0$, we can let $\delta = \frac{\varepsilon}{|m|}$. Then when $|x - y| < \delta$, we have $|(mx + b) - (my + b)| = |mx - my| = |m(x - y)| = |m||x - y| < |m|\delta = |m|\frac{\varepsilon}{|m|} = \varepsilon$. In the case $m = 0$, $|(mx + b) - (my - b)| = 0$, which is alwasy less than ε, so any δ will work.

Problem 8. Let $f(x) = \frac{1}{x}$.

(a) Show that for any $c > 0$, $f(x)$ is uniformly continuous on $[c, \infty)$,

(b) Show that $f(x)$ is not uniformly continuous on $(0, \infty)$.

Answer:

(a) Let $c > 0$. To show that $\frac{1}{x}$ is uniformly continuous on $[c, \infty)$, let $\varepsilon > 0$. We must show that there is a $\delta > 0$ such that for all $x \in [c, \infty)$, if $|x - y| < \delta$, then $\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$. Let $\delta = c^2\varepsilon$. Let $x, y \in [c, \infty)$ with $|x - y| < \delta$. Note that since $x \geq c > 0$, we have $\frac{1}{x} \leq \frac{1}{c}$. Similarly, $\frac{1}{y} \leq \frac{1}{c}$. So, $\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{1}{x} \cdot \frac{1}{y} \cdot |x - y| < \frac{1}{c} \cdot \frac{1}{c} \cdot \delta = \frac{1}{c^2}(c^2\varepsilon) = \varepsilon$.

(b) Letting $\varepsilon = 1$ in the definition of uniform continuity, we must show that for any $\delta > 0$ there exist $x, y \in (0, \infty)$ such that $|x - y| < \delta$ but $\left|\frac{1}{x} - \frac{1}{y}\right| \geq 1$. In the case $\delta \geq 1$, we can let $x = \frac{1}{2}, y = 1$. Then $|x - y| = \frac{1}{2} < \delta$, but $|\frac{1}{x} - \frac{2}{x}| \geq 1$ since it is in fact $|2 - 1| = 1$. In the case $\delta < 1$, let $x = \delta, y = \frac{1}{2}\delta$. Then $|x - y| = |\delta - \frac{1}{2}\delta| = \frac{1}{2}\delta < \delta = 1$, but $\left|\frac{1}{x} - \frac{2}{x}\right| = \left|\frac{1}{\delta} - \frac{2}{\delta}\right| = \frac{1}{\delta} > 1$.

[Easier proof, based on student response: Let $\varepsilon = 1$. Given $\delta > 0$, choose any n with $\frac{1}{n^2} < \delta$. Let $x = \frac{1}{n}, y = \frac{1}{n+1}$. Then $|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n^2+n} < \frac{1}{n^2} < \delta$, and $\left|\frac{1}{x} - \frac{1}{y}\right| = |n - (n + 1)| = 1$, which is not less than ε.]

Problem 9 (Textbook problem 2.6.12ab)**.** We say that a function f satisfies a **Lipschitz condition** if there is a positive real number M such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < M|x - y|$. We say that a function f satisfies a **Lipschitz condition** if there is a positive real number M such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| < M|x - y|$. Show that if f satisfies a Lipschitz condition, then f is uniformly continuous on $(-\infty, \infty)$.

Answer:

Suppose that f satisfies the Lipschitz condition $|f(x) - f(y)| < M|x - y|$ for all $x, y \in \mathbb{R}$. Note that M must be strictly positive. Let $\varepsilon > 0$. We must find $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $\delta = \frac{\varepsilon}{M}$. Then if $|x - y| < \delta$, we have that $|f(x) - f(y)| < M|x - y| < M\delta = M\frac{\varepsilon}{M} = \varepsilon$.