Problem 1. Suppose that \( f(x) \) is defined and bounded on an open interval containing 0, except possibly at 0 itself. (That is, there is a number \( B \) such that \(|f(x)| < B\) for all \( x \) in that interval, except possibly \( x = 0 \).) Show that \( \lim_{x \to 0} xf(x) = 0 \). [Hint: The product rule does not apply here. Use the Squeeze Theorem and the fact that \(|x|\) is a continuous function.]

Answer:

On the open interval where \( f \) is defined, we have that \(|xf(x)| = |x||f(x)| < |x||B|\). (Note that \( B \) must be greater than zero, or else \(|f(x)| < B\) would be impossible.) This inequality is equivalent to \(-B|x| < xf(x) < B|x|\). Since \(|x|\) is a continuous function of \( x \) and any constant multiple of a continuous function is continuous, we know that the functions \(-B|x|\) and \(B|x|\) are both continuous. So, \( \lim_{x \to 0}(-B|x|) = \lim_{x \to 0}B|x| = B|0| = 0 \). Applying the Squeeze Theorem to \(-B|x| < xf(x) < B|x|\), we see that \( \lim_{x \to 0} xf(x) = 0 \).

Problem 2. If \( f(x) \) is a continuous function, then we know that \(|f(x)|\) is also continuous, since it is a composition of continuous functions. Give a counterexample to show that the converse does not hold. That is, find a function \( f(x) \) such that \(|f(x)|\) is continuous, but \( f(x) \) is not continuous.

Answer:

Let \( E(x) = D(x) - \frac{1}{2} \), where \( D(x) \) is the Dirichlet function. That is,

\[
E(x) = \begin{cases} 
1/2 & \text{if } x \text{ is rational} \\
-1/2 & \text{if } x \text{ is irrational}
\end{cases}
\]

\( E(x) \) is not continuous anywhere. But \(|E(x)|\) is the constant function, \(|E(x)| = \frac{1}{2}\), which is continuous everywhere.

For a simpler example, define

\[
f(x) = \begin{cases} 
1 & \text{if } x \neq 0 \\
-1 & \text{if } x = 0
\end{cases}
\]

Then \( f \) is not continuous at 0, but \(|f(x)| = 1\) for all \( x \) and so is continuous.

Problem 3 (Textbook problem 2.5.7). Suppose that \( f \) is continuous at \( a \) and that \( f(a) > 0 \). Prove that there is a \( \delta > 0 \) such that \( f(x) > 0 \) for all \( x \) in the interval \((a - \delta, a + \delta)\).

Answer:

Suppose that \( f \) is continuous at \( x = a \). Let \( \varepsilon = f(a) \), which is greater than zero by assumption. From the definition of continuity, we can find a \( a \) \( \delta > 0 \) such that for any \( x \), if \(|x - a| < \delta\), then \(|f(x) - f(a)| < \varepsilon = f(a)\). This inequality is equivalent to \(-f(a) < f(x) - f(a) < f(a)\). Adding \( f(a) \) to the inequality \(-f(a) < f(x) - f(a) < f(a)\) gives \(0 < f(x)\). So for any \( x \in (a - \delta, a + \delta) \), we have \( f(x) > 0 \).
Problem 4 (Textbook problem 2.4.10). Prove: If \( \lim_{x \to a^+} f(x) = L \) and if \( c(x) \) is a function such that \( a < c(x) < x \) for all \( x \) in some interval \( (a, b) \), then \( \lim_{x \to a^+} f(c(x)) = L \). [Hint: This is confusing but actually easy.]

Answer:

Let \( \varepsilon > 0 \). We must find \( \delta > 0 \) such that for any \( x \), if \( 0 < x - a < \delta \), then \( |f(c(x)) - L| < \varepsilon \). Since \( \lim_{x \to a^+} f(x) = L \), we know that there is a \( \delta \) such that for any \( y \), if \( 0 < y - a < \delta \), then \( |f(y) - L| < \varepsilon (\ast) \). We can take \( \delta \leq b - a \), so that we know that \( a < c(x) < x \) for all \( x \) satisfying \( a < x < a + \delta \).

Using the same \( \delta \), suppose that \( 0 < x - a < \delta \). That is \( a < x < a + \delta \), so we know by our assumption that \( a < c(x) < x \). So we get that \( a < c(x) < x < a + \delta \), which gives \( a < c(x) < a + \delta \) and then \( 0 < c(x) - a < \delta \). Applying \( (\ast) \) with \( y = c(x) \), we get \( |f(c(x)) - L| < \varepsilon \), which is what we needed to show.

Problem 5. Let \( f \) be a continuous function on the interval \( [a, b] \), and suppose that \( f(x) \in \mathbb{Q} \) for all \( x \in [a, b] \). Show that \( f \) is constant on \([a, b]\). [Hint: Use the Intermediate Value Theorem.]

Answer:

Suppose, for the sake of contradiction, that \( f(x) \) is not constant. Then there are points \( x_1 \) and \( x_2 \) in \([a, b]\) such that \( f(x_1) \neq f(x_2) \). Without loss of generality, we can take \( x_1 < x_2 \). Now, \( f \) is continuous on the interval \([x_1, x_2]\), and so satisfies the Intermediate Value Theorem there. Since \( f(x_1) \neq f(x_2) \), we know by the density of the irrational numbers that there is some irrational number \( y \) between \( f(x_1) \) and \( f(x_2) \). By the IVT, there must exist some \( c \in [x_1, x_2] \) such that \( f(c) = y \). But this contradicts the assumption that \( f(x) \in \mathbb{Q} \) for all \( x \in [a, b] \). So, in fact, \( f \) must be constant.

Problem 6 (Textbook problem 2.6.7b). Show that \( p(x) = x^4 - x^3 + x^2 + x - 1 \) has at least two roots in the interval \([-1, 1]\).

Answer:

Since \( p \) is a polynomial, it is continuous everywhere, and the Intermediate Value Theorem will apply to \( p \) on any closed, bounded interval. Note that \( p(-1) = 1, p(0) = -1, \) and \( p(1) = 1 \). Since \( p(-1) > 0 > p(0), \) then by the IVT applied to \( p \) on the interval \([-1, 0]\), \( p(a) = 0 \) for some \( a \in (-1, 0) \). Since \( p(0) < 0 < p(1), \) then by the IVT applied to \( p \) on the interval \([0, 1]\), \( p(b) = 0 \) for some \( b \in (0, 1) \). So \( p \) has at least the roots \( a \) and \( b \) in the interval \([-1, 1]\).

Problem 7. Show that any linear function \( f(x) = mx + b \) is uniformly continuous on \( \mathbb{R} \).

Answer:

Let \( \varepsilon > 0 \). We must find \( \delta > 0 \) such that for all \( x, y \in \mathbb{R}, \) if \( |x - y| < \delta \) it follows that \( |(mx + b) - (my + b)| < \varepsilon \). In the case \( m \neq 0 \), we can let \( \delta = \frac{\varepsilon}{|m|} \). Then when \( |x - y| < \delta \), we have \( |(mx + b) - (my + b)| = |mx - my| = |m(x - y)| = |m||x - y| < |m|\delta = |m|\frac{\varepsilon}{|m|} = \varepsilon \). In the case \( m = 0, |(mx + b) - (my - b)| = 0, \) which is always less than \( \varepsilon \), so any \( \delta \) will work.
Problem 8. Let \( f(x) = \frac{1}{x} \).

(a) Show that for any \( c > 0 \), \( f(x) \) is uniformly continuous on \([c, \infty)\),

(b) Show that \( f(x) \) is not uniformly continuous on \((0, \infty)\).

**Answer:**

(a) Let \( c > 0 \). To show that \( \frac{1}{x} \) is uniformly continuous on \([c, \infty)\), let \( \varepsilon > 0 \). We must show that there is a \( \delta > 0 \) such that for all \( x \in [c, \infty) \), if \( |x - y| < \delta \), then \( \frac{1}{x} - \frac{1}{y} < \varepsilon \). Let \( \delta = c^2 \varepsilon \). Let \( x, y \in [c, \infty) \) with \( |x - y| < \delta \). Note that since \( x \geq c > 0 \), we have \( \frac{1}{x} \leq \frac{1}{c} \).

Similarly, \( \frac{1}{y} \leq \frac{1}{c} \). So, \( \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{y - x}{xy} \right| = \frac{1}{x} \cdot \frac{1}{y} \cdot |x - y| < \frac{1}{c} \cdot \frac{1}{c} \cdot \delta = \frac{1}{c^2} (c^2 \varepsilon) = \varepsilon \).

(b) Letting \( \varepsilon = 1 \) in the definition of uniform continuity, we must show that for any \( \delta > 0 \) there exist \( x, y \in (0, \infty) \) such that \( |x - y| < \delta \) but \( \frac{1}{x} - \frac{1}{y} \geq 1 \). In the case \( \delta \geq 1 \), we can let \( x = \frac{1}{2}, y = 1 \). Then \( |x - y| = \frac{1}{2} < \delta \), but \( \frac{1}{x} - \frac{2}{x} \geq 1 \) since it is in fact \( |2 - 1| = 1 \). In the case \( \delta < 1 \), let \( x = \delta, y = \frac{1}{2} \delta \). Then \( |x - y| = |\delta - \frac{1}{2} \delta| = \frac{1}{2} \delta < \delta = 1 \), but \( \frac{1}{x} - \frac{2}{x} \geq 1 \). \( \frac{1}{x} - \frac{2}{x} = |n - \frac{1}{n} + 1| = 1 \), which is not less than \( \varepsilon \).

Problem 9 (Textbook problem 2.6.12ab). We say that a function \( f \) satisfies a **Lipschitz condition** if there is a positive real number \( M \) such that for all \( x, y \in \mathbb{R} \), \( |f(x) - f(y)| < M|x - y| \). We say that a function \( f \) satisfies a **Lipschitz condition** if there is a positive real number \( M \) such that for all \( x, y \in \mathbb{R} \), \( |f(x) - f(y)| < M|x - y| \). Show that if \( f \) satisfies a Lipschitz condition, then \( f \) is uniformly continuous on \((-\infty, \infty)\).

**Answer:**

Suppose that \( f \) satisfies the Lipschitz condition \( |f(x) - f(y)| < M|x - y| \) for all \( x, y \in \mathbb{R} \). Note that \( M \) must be strictly positive. Let \( \varepsilon > 0 \). We must find \( \delta > 0 \) such that for all \( x, y \in \mathbb{R} \), if \( |x - y| < \delta \), then \( |f(x) - f(y)| < \varepsilon \). Let \( \delta = \frac{\varepsilon}{M} \). Then if \( |x - y| < \delta \), we have that \( |f(x) - f(y)| < M|x - y| < M \delta = M \frac{\varepsilon}{M} = \varepsilon \).