Problem 1. Suppose that \( f(x) \) is defined and bounded on an open interval containing 0, except possibly at 0 itself. (That is, there is a number \( B \) such that \( |f(x)| < B \) for all \( x \) in that interval, except possibly \( x = 0 \).) Show that \( \lim_{x\to0} xf(x) = 0 \). [Hint: The product rule does not apply here. Use the Squeeze Theorem and the fact that \( |x| \) is a continuous function.]

Problem 2. If \( f(x) \) is a continuous function, then we know that \( |f(x)| \) is also continuous, since it is a composition of continuous functions. Give a counterexample to show that the converse does not hold. That is, find a function \( f(x) \) such that \( |f(x)| \) is continuous, but \( f(x) \) is not continuous.

Problem 3 (Textbook problem 2.5.7). Suppose that \( f \) is continuous at \( a \) and that \( f(a) > 0 \). Prove that there is a \( \delta > 0 \) such that \( f(x) > 0 \) for all \( x \) in the interval \( (a-\delta,a+\delta) \).

Problem 4 (Textbook problem 2.4.10). Prove: If \( \lim_{x\to a^+} f(x) = L \) and if \( c(x) \) is a function such that \( a < c(x) < x \) for all \( x \) in some interval \( (a,b) \), then \( \lim_{x\to a^+} f(c(x)) = L \). [Hint: This is confusing but actually easy.]

Problem 5. Let \( f \) be a continuous function on the interval \([a,b]\), and suppose that \( f(x) \in \mathbb{Q} \) for all \( x \in [a,b] \). Show that \( f \) is constant on \([a,b]\). [Hint: Use the Intermediate Value Theorem.]

Problem 6 (Textbook problem 2.6.7b). Show that \( p(x) = x^4 - x^3 + x^2 + x - 1 \) has at least two roots in the interval \([-1, 1]\).

Problem 7. Show that any linear function \( f(x) = mx + b \) is uniformly continuous on \( \mathbb{R} \).

Problem 8. Let \( f(x) = \frac{1}{x} \).
(a) Show that for any \( c > 0 \), \( f(x) \) is uniformly continuous on \([c, \infty)\),
(b) Show that \( f(x) \) is not uniformly continuous on \((0, \infty)\).

Problem 9 (Textbook problem 2.6.12a). We say that a function \( f \) satisfies a Lipschitz condition if there is a positive real number \( M \) such that for all \( x, y \in \mathbb{R} \), \( |f(x) - f(y)| < M|x - y| \). Show that if \( f \) satisfies a Lipschitz condition, then \( f \) is uniformly continuous on \((-\infty, \infty)\).