Problem 1. Let \((M,d)\) be a metric space, and let \(A\) be a subset of \(M\). Prove that \(A\) is open if and only if \(A\) is equal to a union of open balls.

Answer:

\[ \implies \) Suppose that \(A\) is an open set in the metric space \((M,d)\). Then for any \(x \in A\), there is an \(\varepsilon_x > 0\) such that \(B_{\varepsilon_x}(x) \subseteq A\). We claim that \(\bigcup_{x \in A} B_{\varepsilon_x}(x) = A\). But since \(B_{\varepsilon_x}(x) \subseteq A\), it is also true that \(\bigcup_{x \in A} B_{\varepsilon_x}(x) \subseteq A\). And for \(a \in A\), we have \(a \in B_{\varepsilon_x}(a) \subseteq \bigcup_{x \in A} B_{\varepsilon_x}(x)\), so \(A \subseteq \bigcup_{x \in A} B_{\varepsilon_x}(x)\).

\[ \Longleftarrow \) Suppose that \(A\) is a union of open balls, say \(A = \bigcup_{\alpha \in B} B_{\varepsilon_\alpha}(x_\alpha)\). Since \(B_{\varepsilon_\alpha}(x_\alpha)\) is an open set, \(A\) is a union of open sets, and we know that any union of open sets is open.

Problem 2. Consider the metric space \((\mathbb{R},d)\), where \(d\) is the usual metric on \(\mathbb{R}\). For each \(n = 1, 2, 3, \ldots\), let \(O_n\) be the open set \(O_n = (1, 1 + \frac{1}{n})\). Show that \(\{O_n\}_{n=1}^\infty\) is an infinite collection of open sets whose intersection is not open. And find an infinite collection of closed subsets of \((\mathbb{R},d)\) whose union is not closed.

Answer:

In fact, the intersection of all the \(O_n\) is the empty set, which is open. So the problem is incorrect.

If \(O_n = (1 - \frac{1}{n}, 1 + \frac{1}{n})\), then \(\bigcap_{n=1}^\infty O_n = \{1\}\), because 1 is a member of each of the intervals, and for any number \(x > 1\), there is an \(n \in \mathbb{N}\) such that \(\frac{1}{n} < x - 1\), and therefore \(1 + \frac{1}{n} < x\) and \(x \not\in (1 - \frac{1}{n}, 1 + \frac{1}{n})\), and similarly, if \(x < 0\), then \(x\) is not in the intersection. The set \(\{1\}\) is not open because it does not contain any open ball about 1.

For another example, if \(O_n = (0, 1 + \frac{1}{n})\), then the intersection is \((0,1]\), which is not open.

For an infinite collection of closed sets whose union is not closed, we can use \(\bigcup_{x \in (0,1)} \{x\}\).

We have shown that any singleton set, \(\{x\}\), is closed. This union is clearly equal to the interval \((0, 1]\), which is not closed since it does not include 1, which is an accumulation point of the set. (For a more traditional example, use \(\bigcup_{n=1}^\infty [0,1 - \frac{1}{n}]\). This union is equal to \([0,1)\), which is not closed.)

Problem 3. Let \(X\) be any non-empty, bounded subset of \(\mathbb{R}\), and let \(\lambda\) be the least upper bound of \(X\). Show that \(\lambda \in \bar{X}\). That is, the least upper bound of any set is an element of the closure of that set. [Hint: Use the definition of closure of \(X\) as the set of all points of \(X\) plus all accumulation points of \(X\), and use Problem 1 from Homework 3.]
Problem 4. Let \((A, \sigma), (B, \tau), \) and \((C, \eta)\) be metric spaces. Let \(f: A \to B\) and \(g: B \to C\). Suppose that \(f\) and \(g\) are continuous functions. Prove that their composition, \(g \circ f\), is a continuous function.

Answer:

Let \((A, \sigma), (B, \tau)\), and \((C, \eta)\) be metric spaces. Let \(f: A \to B\) and \(g: B \to C\). Suppose that \(f\) and \(g\) are continuous functions. We want to show that \(g \circ f\) is continuous. Let \(a \in A\), and let \(\varepsilon > 0\). We want to find \(\delta > 0\) such that \(B_\delta(a) \subseteq B_\varepsilon(g(f(a))).\)

Since \(g\) is continuous at \(f(a)\), there is a \(\eta > 0\) such that \(B_\eta(f(a)) \subseteq B_\varepsilon(g(f(a))).\) Since \(f\) is continuous at \(a\), there is a \(\delta > 0\) such that \(B_\delta(a) \subseteq B_\eta(f(a)).\) So, we have that \(B_\delta(a) \subseteq B_\eta(f(a)) \subseteq B_\varepsilon(g(f(a))),\) and therefore for this \(\delta\), \(B_\delta(a) \subseteq B_\varepsilon(g(f(a))).\)

(Alternative proof: Let \(O\) be open in \(C\). We want to show \((g \circ f)^{-1}(O)\) is open in \(A\). Now, \((g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)).\) Since \(g\) is continuous, \(g^{-1}(O)\) is open in \(B\). Since \(f\) is continuous, \(f^{-1}(g^{-1}(O))\) is open in \(A\). So we are done.)

(Another alternative proof: Let \(\{x_n\}_{n=1}^\infty\) converge to \(a\) in \(A\). We want to show \(\{g(f(x_n))\}_{n=1}^\infty\) converges in \(C\) to \(g(f(a))\). Since \(f\) is continuous, \(\{f(x_n)\}_{n=1}^\infty\) converges to \(f(a)\) in \(B\). Then since \(g\) is continuous, \(\{g(f(x_n))\}_{n=1}^\infty\) converges to \(g(f(a))\) in \(C\). So we are done.)

Problem 5. Let \(\{x_n\}_{n=1}^\infty\) be a convergent sequence in a metric space. Show that its limit is unique. That is, prove the following statement: if \(\{x_n\}_{n=1}^\infty\) converges to \(y\) and \(\{x_n\}_{n=1}^\infty\) converges to \(z\), then \(y = z\).

Answer:

Suppose that \(\lim_{n \to \infty} x_n = x\) and also \(\lim_{n \to \infty} x_n = y\). We want to show \(x = y\). Suppose, for the sake of contradiction, that \(x \neq y\).

Let \(\varepsilon = d(x, y)/2\), which is greater than zero since \(x \neq y\). There is an \(N_1 \in \mathbb{N}\) such that for \(n \geq N_1\), \(d(x_n, x) < \varepsilon\). And there is an \(N_2 \in \mathbb{N}\) such that for \(n \geq N_2\), \(d(x_n, y) < \varepsilon\). Let \(N = \max(N_1, N_2)\). We then have both \(d(x_N, x) < \varepsilon\) and \(d(x_N, y) < \varepsilon\). But that means that

\[
    d(x, y) \leq d(x, x_N) + d(x_N, y) < \varepsilon + \varepsilon = d(x, y)/2 + d(x, y)/2 = d(x, y)
\]

The contradiction \(d(x, y) < d(x, y)\) proves that \(x \neq y\) cannot be the case.

(Alternative direct proof: Show that \(d(x, y) < \varepsilon\) for all \(\varepsilon > 0\), which will prove \(d(x, y) = 0\) and hence \(x = y\). Let \(\varepsilon > 0\). There is an \(N_1 \in \mathbb{N}\) such that for \(n \geq N_1\), \(d(x_n, x) < \frac{\varepsilon}{2}\). And there is an \(N_2 \in \mathbb{N}\) such that for \(n \geq N_2\), \(d(x_n, y) < \frac{\varepsilon}{2}\). Let \(N = \max(N_1, N_2)\). Then \(d(x, y) \leq d(x, x_N) + d(x_N, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\).)