Problem 1. Let (M, d) be a metric space, and let $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{R}$ be two functions from M to $\mathbb{R}$ (where $\mathbb{R}$ has its usual metric). Let $a \in M$. Suppose f and g are continuous at a. Show that the function $f + g$ is continuous at a, where $(f + g)(x) = f(x) + g(x)$ for $x \in M$. [Hint: Just imitate the proof for functions from $\mathbb{R}$ to $\mathbb{R}$.]

Answer:

Let $\varepsilon > 0$. Since f is continuous at a, there is a $\delta_1 > 0$ such that for all $x \in B^d_{\delta_1}(a)$, $|f(x) - f(a)| < \frac{\varepsilon}{2}$. And since f is continuous at a, there is a $\delta_2 > 0$ such that for all $x \in B^d_{\delta_2}(a)$, $|g(x) - g(a)| < \frac{\varepsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. We want to show that for all $x \in B^d_\delta(a)$, $|(f+g)(x) - (f+g)(a)| < \varepsilon$. Let $x \in B^d_\delta(a)$. Since $\delta \le \delta_1$ and $\delta \le \delta_2$, we have both $x \in B^d_{\delta_1}(a)$ and $x \in B^d_{\delta_2}(a)$. From that, we get both $|f(x) - f(a)| < \frac{\varepsilon}{2}$ and $|g(x) - g(a)| < \frac{\varepsilon}{2}$, and therefore,

$$\begin{aligned}
|(f+g)(x) - (f+g)(a)| &= |f(x) + g(x) - f(a) - g(a)| \\
&= |(f(x) - f(a)) + (g(x) - g(a))| \\
&\le |f(x) - f(a)| + |g(x) - g(a)| \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Problem 2. Let X be any set. Consider the metric space (X, δ) where δ is the discrete metric, $\delta\colon X \times X \to \mathbb{R}$ by $\delta(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \ne b \end{cases}$. Suppose that $\{x_i\}_{i=1}^\infty$ is a **convergent** sequence in the metric space (X, δ). Show that there is a number N such that $x_N = x_{N+1} = x_{N+2} = \cdots$. (We say that the sequence is "eventually constant.")

Answer:

Suppose that the sequence $\{x_n\}_{n=1}^\infty$ converges to z. Let $\varepsilon = \frac{1}{2}$. Since $\lim\limits_{n\to\infty} x_n = z$, there is a $N \in \mathbb{N}$ such that for all $n \ge N$, $x_n \in B_\varepsilon(z)$; that is, $x_n \in B_{1/2}(z)$. But in the discrete metrix, every point other than z is at distance 1 from z, so $B_{1/2}(z) = \{z\}$. Since $x_n \in B_{1/2}(z)$ for $n \ge N$, it must be that $x_n = z$ for $n \ge N$.

Problem 3. Let (M, d) be a metric space and let $X \subseteq M$. The closure, $\overline{X}$ of X can be defined as the set containing all the points of X plus all the accumulation points of X. Show that the closure of X can be characterized as follows: For $z \in M$, $z \in \overline{X}$ if and only if there is a sequence $\{x_n\}_{n=1}^\infty$ of points of X such that $\lim\limits_{n\to\infty} x_n = z$. [Hint: Treat separately the cases where $z \in X$ and where z is an accumulation point of X.]

Answer:

$\Longrightarrow$) Let $z \in \overline{X}$. We want to find a sequence, $\{x_n\}_{n=1}^\infty$, of points of X that convereges to z. Since $z \in \overline{X}$, we know that $z \in X$ or z is an accumulation point of X.

Consider the case $z \in X$. In that case, we can let $x_n = z$ for all n, since the constant sequence $\{z\}_{n=1}^{\infty}$ is a sequence of points of X that converges to z.

Now consider the case z is an accumulation point of z. Then, for any $\varepsilon > 0$, we know that there is some $a \in X$ such that $d(z, a) < \varepsilon$. For $n \in \mathbb{N}$, let x_n be a point of X such that $d(z, x_n) < \frac{1}{n}$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of X that converges to z. (To prove convergence, let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, by Archimedes' Principle. So for $n \geq N$, we have $d(x_n, z) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. That is, for $n \geq N$, $x_n \in B_\varepsilon(z)$.)

$\Longleftarrow$) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of X that coverges to $z \in M$. We want to show that $z \in \overline{X}$.

If there is an $N \in \mathbb{N}$ such that $x_n = z$ for all $n > N$, then $z \in X$ because all elements of the sequence are in X, and $z \in \overline{X}$ because $X \subseteq \overline{X}$.

Suppose that no such N exists. We show that in that case, z is an accumulation point of X, which implies $z \in \overline{X}$. To show z is an accumulation point, let $\varepsilon > 0$. We must find an $a \in X$ such that $d(z, a) < \varepsilon$ and $a \notin X$. Since $\lim\limits_{n\to\infty} x_n = z$, there is an $N \in \mathbb{N}$ such that $d(x_n, z) < \varepsilon$ for all $n \geq N$. But we have assumed that the sequence is not eventually constant, so there must be some $n_o > N$ for which $x_{n_o} \neq z$. Let a be that x_{n_o}. Then we know $d(a, z) < \varepsilon$, and we know $a \in X$ since all terms of the sequence are in X.

(Improved proof for $\Longleftarrow$: Suppose thatat $\{x_n\}_{n=1}^{\infty}$ is a sequence of points of X that coverges to $z \in M$. We want to show that $z \in \overline{X}$.

If $z \in X$, there is nothing to prove, since $X \subseteq \overline{X}$ and so $z \in X$ implies $x \in \overline{X}$.

So suppose $z \notin X$. We show z is an accumulation point of X, which means it is in $\overline{X}$. To show z is an accumulation point, let $\varepsilon > 0$. We must find an $x \in X$ such that $0 < d(x, z) < \varepsilon$. Since all of the elements of the sequence are in X, and z is not in X, we know that for all n, $x_n \neq z$ and $d(x_n, z) > 0$. And since the sequence converges to z, there must be an x_N such that $d(x_N, z) < \varepsilon$. Since $x_N \in X$, we have found an element x_N of X such that $0 < d(z, x_N) < \varepsilon$.)

Problem 4 (From textbook problem 3.1.3a)**.** Even though $|x|$ is not differentiable at 0, show that the function $g(x) = \frac{1}{2}x|x|$ is differentiable at 0, and show that $g'(x) = |x|$ for all x. (Thus, $|x|$ has antiderivative $\frac{1}{2}x|x|$.)

Answer:

For $a < 0$, $g'(a) = \frac{d}{dx}\Big|_{x=a} \frac{1}{2}x|x| = \frac{d}{dx}\Big|_{x=a} \frac{1}{2}x(-x) = \frac{d}{dx}\Big|_{x=a}\big(-\frac{1}{2}x^2\big) = -\frac{1}{2} \cdot 2a = -a = |a|$. So $g'(x) = |x|$ in the case $a < 0$.

For $a > 0$, $g'(a) = \frac{d}{dx}\Big|_{x=a} \frac{1}{2}x|x| = \frac{d}{dx}\Big|_{x=a} \frac{1}{2}x(x) = \frac{d}{dx}\Big|_{x=a}\big(\frac{1}{2}x^2\big) = \frac{1}{2} \cdot 2a = a = |a|$. So $g'(x) = |x|$ in the case $a > 0$.

Finally, for the case $a = 0$, we must show that $g'(0) = \frac{1}{2}0|0|$, that is $g'(0) = 0$. But in this case we can calculate

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\frac{1}{2}x|x| - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{\frac{1}{2}x|x|}{x} \\ &= \lim_{x\to 0} \tfrac{1}{2}|x| \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}|0|, \text{ since } |x| \text{ is continuous at } 0 \\
&= 0
\end{aligned}
$$

Problem 5 (Textbook problem 3.3.10)**.** A **fixed point** of a function is a point d such that $f(d) = d$. Suppose that f is differentiable everywhere and that $f'(x) < 1$ for all x. Show that there can be at most one fixed point for f. [Hint: Suppose that a and b are two fixed points of f. Apply the Mean Value Theorem to obtain a contradiction.]

Answer:

Let f be differentiable everywhere and suppose that $f'(x) < 1$ for all x. We want to show that f has at most one fixed point. Suppose, for the sake of contradiction, that f has more than one fixed point. Let a and b be distinct fixed points of f. That is $a \neq b$, and $f(a) = a$, and $f(b) = b$. By the Mean Value Theorem, there is a c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. But $f(b) = b$ and $f(a) = a$, so $f'(c) = \frac{b-a}{b-a} = 1$. But that contradicts the assumption that $f'(x) < 1$ for all x.

Problem 6 (From textbook problem 3.3.2)**.** Recall that f satisfies a Lipschitz condition if there is a constant M such that $|f(b) - f(a)| \leq M|b - a|$ for all a, b. Problem #9 on Homework #4 proved that any function that satisfies a Libschitz condition is uniformly continuous. Let f be a function that is differentiable on some interval I (not necessarily bouned or closed), and suppose $|f'(x)| \leq M$ for all x, where M is some constant. Use the Mean Value Theorem to prove that $|f(b) - f(a)| \leq M|a - b|$ for all a, b. Conclude that f is uniformly continuous.

Answer:

Suppose f is differentiable on an interval I and $|f'(x)| < M$ for all $x \in I$. Let a and b be distinct points in I, where we can assume witout loss of generality that $a < b$. Since I is an interval, it contains the entire interval $[a, b]$. Since f is continuous on $[a, b]$ and differentiable on (a, b), the Mean Value Theorem applies. That is, there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Since $c \in I$, we know by assumption that $|f'(c)| \leq M$. That is, $\left|\frac{f(b)-f(a)}{b-a}\right| < M$. Since $|b - a| > 0$, this implies $|f(b) - f(a)| \leq M|b - a|$. That is, f satisfies a Lipschitz condition on the interval I with Lipshitz constant M. We conclude by Problem #9 on Homework #4 that f is uniformly continuous on I.

(Note: The problem originally assumed, incorrectly, only that $f'(c) < M$. If we only have $f'(c) < M$, that leaves open the possibility that $f'(c)$ is some large negative number, and in that case the statement is not true. For example, if $f(x) = \frac{1}{x}$ for $x > 0$, then $f'(c) < 1$ for all $c > 0$, but f is not uniformly continuous on $(0, \infty)$. So, we assume that $|f'(c)| < M$ (which implies $M > 0$).)