**Problem 1.** Let \((M, d)\) be a metric space, and let \(f : M \to \mathbb{R}\) and \(g : M \to \mathbb{R}\) be two functions from \(M\) to \(\mathbb{R}\) (where \(\mathbb{R}\) has its usual metric). Let \(a \in M\). Suppose \(f\) and \(g\) are continuous at \(a\). Show that the function \(f + g\) is continuous at \(a\), where \((f + g)(x) = f(x) + g(x)\) for \(x \in M\). [Hint: Just imitate the proof for functions from \(\mathbb{R}\) to \(\mathbb{R}\).]

**Answer:**

Let \(\varepsilon > 0\). Since \(f\) is continuous at \(a\), there is a \(\delta_1 > 0\) such that for all \(x \in B^d_{\delta_1}(a)\), \(|f(x) - f(a)| < \frac{\varepsilon}{2}\). And since \(f\) is continuous at \(a\), there is a \(\delta_2 > 0\) such that for all \(x \in B^d_{\delta_2}(a)\), \(|g(x) - g(a)| < \frac{\varepsilon}{2}\).

Let \(\delta = \min(\delta_1, \delta_2)\). We want to show that for all \(x \in B^d_{\delta}(a)\), \(|(f + g)(x) - (f + g)(a)| < \varepsilon\). Let \(x \in B^d_{\delta}(a)\). Since \(\delta \leq \delta_1\) and \(\delta \leq \delta_2\), we have both \(x \in B^d_{\delta_1}(a)\) and \(x \in B^d_{\delta_2}(a)\). From that, we get both \(|f(x) - f(a)| < \frac{\varepsilon}{2}\) and \(|g(x) - g(a)| < \frac{\varepsilon}{2}\), and therefore,

\[
|(f + g)(x) - (f + g)(a)| = |f(x) + g(x) - f(a) - g(a)|
= |(f(x) - f(a)) + (g(x) - g(a))|
\leq |f(x) - f(a)| + |g(x) - g(a)|
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
= \varepsilon
\]

**Problem 2.** Let \(X\) be any set. Consider the metric space \((X, \delta)\) where \(\delta\) is the discrete metric, \(\delta : X \times X \to \mathbb{R}\) by \(\delta(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}\). Suppose that \(\{x_i\}_{i=1}^{\infty}\) is a convergent sequence in the metric space \((X, \delta)\). Show that there is a number \(N\) such that \(x_N = x_{N+1} = x_{N+2} = \cdots\). (We say that the sequence is “eventually constant.”)

**Answer:**

Suppose that the sequence \(\{x_n\}_{n=1}^{\infty}\) converges to \(z\). Let \(\varepsilon = \frac{1}{2}\). Since \(\lim_{n \to \infty} x_n = z\), there is a \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(x_n \in B_\varepsilon(z)\); that is, \(x_n \in B_{1/2}(z)\). But in the discrete metric, every point other than \(z\) is at distance 1 from \(z\), so \(B_{1/2}(z) = \{z\}\). Since \(x_n \in B_{1/2}(z)\) for \(n \geq N\), it must be that \(x_n = z\) for \(n \geq N\).

**Problem 3.** Let \((M, d)\) be a metric space and let \(X \subseteq M\). The closure, \(\overline{X}\) of \(X\) can be defined as the set containing all the points of \(X\) plus all the accumulation points of \(X\). Show that the closure of \(X\) can be characterized as follows: For \(z \in M\), \(z \in \overline{X}\) if and only if there is a sequence \(\{x_n\}_{n=1}^{\infty}\) of points of \(X\) such that \(\lim_{n \to \infty} x_n = z\). [Hint: Treat separately the cases where \(z \in X\) and where \(z\) is an accumulation point of \(X\).]

**Answer:**

\(\implies\) Let \(z \in \overline{X}\). We want to find a sequence, \(\{x_n\}_{n=1}^{\infty}\), of points of \(X\) that converges to \(z\). Since \(z \in \overline{X}\), we know that \(z \in X\) or \(z\) is an accumulation point of \(X\).
Consider the case $z \in X$. In that case, we can let $x_n = z$ for all $n$, since the constant sequence $(z)_{n=1}^\infty$ is a sequence of points of $X$ that converges to $z$.

Now consider the case $z$ is an accumulation point of $z$. Then, for any $\varepsilon > 0$, we know that there is some $a \in X$ such that $d(z, a) < \varepsilon$. For $n \in \mathbb{N}$, let $x_n$ be a point of $X$ such that $d(z, x_n) < \frac{1}{n}$. Then the sequence $(x_n)_{n=1}^\infty$ is a sequence of points of $X$ that converges to $z$. (To prove convergence, let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, by Archimedes’ Principle. So for $n \geq N$, we have $d(x_n, z) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. That is, for $n \geq N$, $x_n \in B_\varepsilon(z).$)

$\iff$ Suppose that $(x_n)_{n=1}^\infty$ is a sequence of points of $X$ that converges to $z \in M$. We want to show that $z \in \overline{X}$.

If there is an $N \in \mathbb{N}$ such that $x_n = z$ for all $n > N$, then $z \in X$ because all elements of the sequence are in $X$, and $z \in \overline{X}$ because $X \subseteq \overline{X}$.

Suppose that no such $N$ exists. We show that in that case, $z$ is an accumulation point of $X$, which implies $z \in \overline{X}$. To show $z$ is an accumulation point, let $\varepsilon > 0$. We must find an $a \in X$ such that $d(z, a) < \varepsilon$ and $a \not\in X$. Since $\lim_{n \to \infty} x_n = z$, there is an $N \in \mathbb{N}$ such that $d(x_n, z) < \varepsilon$ for all $n \geq N$. But we have assumed that the sequence is not eventually constant, so there must be some $n_o > N$ for which $x_{n_o} \neq z$. Let $a$ be that $x_{n_o}$. Then we know $d(a, z) < \varepsilon$, and we know $a \in X$ since all terms of the sequence are in $X$.

(Improved proof for $\iff$: Suppose that $(x_n)_{n=1}^\infty$ is a sequence of points of $X$ that converges to $z \in M$. We want to show that $z \in \overline{X}$.

If $z \in X$, there is nothing to prove, since $X \subseteq \overline{X}$ and so $z \in X$ implies $x \in \overline{X}$.

So suppose $z \not\in X$. We show $z$ is an accumulation point of $X$, which means it is in $\overline{X}$. To show $z$ is an accumulation point, let $\varepsilon > 0$. We must find an $a \in X$ such that $0 < d(x, z) < \varepsilon$. Since all of the elements of the sequence are in $X$, and $z$ is not in $X$, we know that for all $n$, $x_n \neq z$ and $d(x_n, z) > 0$. And since the sequence converges to $z$, there must be an $x_N$ such that $d(x_N, z) < \varepsilon$. Since $x_N \in X$, we have found an element $x_N$ of $X$ such that $0 < d(z, x_N) < \varepsilon$.)

**Problem 4** (From textbook problem 3.1.3a). Even though $|x|$ is not differentiable at 0, show that the function $g(x) = \frac{1}{2}x|x|$ is differentiable at 0, and show that $g'(x) = |x|$ for all $x$. (Thus, $|x|$ has antiderivative $\frac{1}{2}x|x|$.)

**Answer:**

For $a < 0$, $g'(a) = \frac{d}{dx} \big|_{x=a} \frac{1}{2}x|x| = \frac{d}{dx} \big|_{x=a} \frac{1}{2}x(-x) = \frac{d}{dx} \big|_{x=a} (-\frac{1}{2}x^2) = -\frac{1}{2} \cdot 2a = -a = |a|$. So $g'(x) = |x|$ in the case $a < 0$.

For $a > 0$, $g'(a) = \frac{d}{dx} \big|_{x=a} \frac{1}{2}x|x| = \frac{d}{dx} \big|_{x=a} \frac{1}{2}x(x) = \frac{d}{dx} \big|_{x=a} (\frac{1}{2}x^2) = \frac{1}{2} \cdot 2a = a = |a|$. So $g'(x) = |x|$ in the case $a > 0$.

Finally, for the case $a = 0$, we must show that $g'(0) = \frac{1}{2}0|0|$, that is $g'(0) = 0$. But in this case we can calculate

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x|x| - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x|x|}{x}$$

$$= \lim_{x \to 0} \frac{1}{2}|x|$$
\[
= \frac{1}{2} |0|, \text{ since } |x| \text{ is continuous at } 0 \\
= 0
\]

**Problem 5** (Textbook problem 3.3.10). A **fixed point** of a function is a point \( d \) such that \( f(d) = d \). Suppose that \( f \) is differentiable everywhere and that \( f'(x) < 1 \) for all \( x \). Show that there can be at most one fixed point for \( f \). [Hint: Suppose that \( a \) and \( b \) are two fixed points of \( f \). Apply the Mean Value Theorem to obtain a contradiction.]

**Answer:**

Let \( f \) be differentiable everywhere and suppose that \( f'(x) < 1 \) for all \( x \). We want to show that \( f \) has at most one fixed point. Suppose, for the sake of contradiction, that \( f \) has more than one fixed point. Let \( a \) and \( b \) be distinct fixed points of \( f \). That is, \( a \neq b \), and \( f(a) = a \) and \( f(b) = b \). By the Mean Value Theorem, there is a \( c \) between \( a \) and \( b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \). But \( f(b) = b \) and \( f(a) = a \), so \( f'(c) = \frac{b-a}{b-a} = 1 \). But that contradicts the assumption that \( f'(x) < 1 \) for all \( x \).

**Problem 6** (From textbook problem 3.3.2). Recall that \( f \) satisfies a Lipschitz condition if there is a constant \( M \) such that \(|f(b) - f(a)| \leq M|b - a|\) for all \( a, b \). Problem #9 on Homework #4 proved that any function that satisfies a Lipschitz condition is uniformly continuous. Let \( f \) be a function that is differentiable on some interval \( I \) (not necessarily bounded or closed), and suppose \(|f'(x)| \leq M \) for all \( x \), where \( M \) is some constant. Use the Mean Value Theorem to prove that \(|f(b) - f(a)| \leq M|a - b|\) for all \( a, b \). Conclude that \( f \) is uniformly continuous.

**Answer:**

Suppose \( f \) is differentiable on an interval \( I \) and \(|f'(x)| < M \) for all \( x \in I \). Let \( a \) and \( b \) be distinct points in \( I \), where we can assume without loss of generality that \( a < b \). Since \( I \) is an interval, it contains the entire interval \([a, b]\). Since \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), the Mean Value Theorem applies. That is, there is a \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \). Since \( c \in I \), we know by assumption that \(|f'(c)| \leq M \). That is, \(|f(b) - f(a)| \leq M|b - a| \). Since \(|b - a| > 0\), this implies \(|f(b) - f(a)| \leq M|b - a| \). That is, \( f \) satisfies a Lipschitz condition on the interval \( I \) with Lipschitz constant \( M \). We conclude by Problem #9 on Homework #4 that \( f \) is uniformly continuous on \( I \).

(Note: The problem originally assumed, incorrectly, only that \( f'(c) < M \). If we only have \( f'(c) < M \), that leaves open the possibility that \( f'(c) \) is some large negative number, and in that case the statement is not true. For example, if \( f(x) = \frac{1}{x} \) for \( x > 0 \), then \( f'(c) < 1 \) for all \( c > 0 \), but \( f \) is not uniformly continuous on \((0, \infty)\). So, we assume that \(|f'(c)| < M \) (which implies \( M > 0 \)).)