This homework is due on Monday, October 17.

Problem 1. Let (M, d) be a metric space, and let $f\colon M \to \mathbb{R}$ and $g\colon M \to \mathbb{R}$ be two functions from M to $\mathbb{R}$ (where $\mathbb{R}$ has its usual metric). Let $a \in M$. Suppose f and g are continuous at a. Show that the function $f + g$ is continuous at a, where $(f + g)(x) = f(x) + g(x)$ for $x \in M$. [Hint: Just imitate the proof for functions from $\mathbb{R}$ to $\mathbb{R}$.]

Problem 2. Let X be any set. Consider the metric space (X, δ) where δ is the discrete metric, $\delta\colon X \times X \to \mathbb{R}$ by $\delta(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases}$. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a **convergent** sequence in the metric space (X, δ). Show that there is a number N such that $x_N = x_{N+1} = x_{N+2} = \cdots$. (We say that the sequence is "eventually constant.") [Hint: The number is the limit of the sequence.]

Problem 3. Let (M, d) be a metric space and let $X \subseteq M$. The closure, $\overline{X}$ of X can be defined as the set containing all the points of X plus all the accumulation points of X. Show that the closure of X can be characterized as follows: For $z \in M$, $z \in \overline{X}$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of X such that $\lim_{n\to\infty} x_n = z$. [Hint: Treat separately the cases where $z \in X$ and where z is an accumulation point of X.]

Problem 4 (From textbook problem 3.1.3a)**.** Even though $|x|$ is not differentiable at 0, show that the function $g(x) = \frac{1}{2}x|x|$ is differentiable at 0, and show that $g'(x) = |x|$ for all x. (Thus, $|x|$ has antiderivative $\frac{1}{2}x|x|$.)

Problem 5 (Textbook problem 3.3.10)**.** A **fixed point** of a function is a point d such that $f(d) = d$. Suppose that f is differentiable everywhere and that $f'(x) < 1$ for all x. Show that there can be at most one fixed point for f. [Hint: Suppose that a and b are two fixed points of f. Apply the Mean Value Theorem to obtain a contradiction.]

Problem 6 (From textbook problem 3.3.2)**.** Recall that f satisfies a Lipschitz condition if there is a constant M such that $|f(b) - f(a)| \leq M|b - a|$ for all a, b. Problem #9 on Homework #4 proved that any function that satisfies a Libschitz condition is uniformly continuous. Let f be a function that is differentiable on some interval I (not necessarily bouned or closed), and suppose $f'(x) \leq M$ for all x, where M is some constant. Use the Mean Value Theorem to prove that $|f(b) - f(a)| \leq M|a - b|$ for all a, b. Conclude that f is uniformly continuous.