Problem 1. We showed that if f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$. Now, suppose we know that $|g|$ is integrable on $[a, b]$. Is it necessarily true that g is integrable on $[a, b]$? [Hint: Consider a simple modification of the Dirichlet function.]

Answer:

It is **not** necessarily true that if $|g|$ is integrable, then g is integrable. Recall that the Dirichlet function is defined as $D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. We know that $D(x)$ is not integrable on $[0, 1]$. Define $g(x) = D(x) - \frac{1}{2} = \begin{cases} 1/2 & \text{if } x \text{ is rational} \\ -1/2 & \text{if } x \text{ is irrational} \end{cases}$. Then $|g|$ is the constant function $\frac{1}{2}$, so $|g|$ is integrable. However, g is not integrable, since if it were then $D(x) = g(x) + \frac{1}{2}$ would also be integrable.

Problem 2. Suppose f is a continuous function on $[a, b]$ and $f(x) > 0$ for $x \in [a, b]$. Define $F(x) = \int_a^x f$. Prove that F is strictly increasing on $[a, b]$. [Hint: This is trivial, using two facts that we have proved.]

Answer:

By the Second Fundamental Theorem of Calculus, $F'(x) = f(x)$ for all $x \in [a, b]$. So saying $f(x) > 0$ means $F'(x) > 0$. By a corollary to the Mean Value Theorem, F is strictly increasing.

[Or, to prove it directly, suppose $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We must show $F(x_1) < F(x_2)$. By the MVT, there is a $c \in [x_1, x_2]$ such that $F'(c) = \frac{F(x_2)-F(x_1)}{x_2-x_1}$. Since $F'(c) > 0$, we have $\frac{F(x_2)-F(x_1)}{x_2-x_1} > 0$. And then, since $x_2 - x_1 > 0$, we can multiply that inequaltity by $x_2 - x_1$ to get $F(x_2) - F(x_1) > 0$. That is, $F(x_2) > F(x_1$.]

[Or, for a different proof, let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. We must show $F(x_1) < F(x_2)$. But $F(x_1) < F(x_2) = \int_{x_1}^{x_2} f > 0$ using the fact that f is continuous and the result in Problem 4c below.]

Problem 3 (Textbook problem 3.4.11)**.** Assume that f is integrable on $[a, b]$. Suppose that J is a real number such that $L(f, P) \leq J \leq U(f, P)$ for every partition P of $[a, b]$. Show that $J = \int_a^b f$. [Hint: Use properties of *sup* and *inf*, that is of lub and glb, and the definition of integrable.]

Answer:

Since f is integrable on $[a, b]$, we know that $\int_a^b f = \sup_P L(f, P) = \inf_P U(f, P)$.

Since $J \geq L(f, P)$ for every partition P of $[a, b]$, we see that J is an upper bound for the set $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$. So, J is greater than or equal to the sup of this set, which is $\int_a^b f$. That is, $J \geq \int_a^b f$.

Since $J \leq U(f, P)$ for every partition P of $[a, b]$, we see that J is a lower bound for the set $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$. So, J is less than or equal to the inf of this set, which is $\int_a^b f$. That is, $J \leq \int_a^b f$.

So we have $\int_a^b f \leq J \leq \int_a^b f$, which means $J = \int_a^b f$.

Problem 4. Prove the following statements.

(a) Assume that f is an integrable function on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$. Prove directly, using the definition of the integral, that $\int_a^b f \geq 0$.

(b) Assume that f and g are integrable on $[a, b]$ and that $f(x) \geq g(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f \geq \int_a^b g$, using part (a) and the linearity of the integral (Theorems 3.5.6 and 3.5.7).

(c) Assume that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a, b]$, and that $f(c) > 0$, where c is some number in (a, b). Show that $\int_a^b f > 0$. [Hint: A previous homework problem already showed that there is a $\delta > 0$ such that $f(x) > \frac{f(c)}{2}$ for all $x \in (c - \delta, c + \delta)$.]

Answer:

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$, and let $M_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}$, as usual. Since $f(x) \geq 0$ for all $x \in [x_{i-1}, x_i]$, we know that $M_i \geq 0$. Therefore $U(f, P)$, which is $\sum_{i=1}^n M_i \cdot (x_i - x_{i-1})$, is a sum of non-negative terms. So $U(f, P) \geq 0$. Since this is true for all partitions of $[a, b]$, zero is a lower bound for the set $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$, which implies that $\inf_P U(f, P) \geq 0$. But $\inf_P U(f, P) = \int_a^b f$, so we have $\int_a^b f \geq 0$.

(b) We know $f(x) \geq g(x)$, and therefore $f(x) - g(x) \geq 0$, for all $x \in [a, b]$. By part (a), $\int_a^b (f - g) \geq 0$. By the linearity of the integral, $\int_a^b (f - g) = \big(\int_a^b f\big) - \big(\int_a^b g\big)$. Combining these facts, $\big(\int_a^b f\big) - \big(\int_a^b g\big) \geq 0$, and therefore $\int_a^b f \geq \int_a^b g$.

(c) By continuity of f at c, there is a $\delta > 0$ such that $f(x) > \frac{f(c)}{2}$ for all $x \in (c - \delta, c + \delta)$. [For the proof: Since $\frac{f(c)}{2} > 0$ and f is continuous at c, there is a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \frac{f(c)}{2}$. That is, for $x \in (c - \delta, c + \delta)$, we get $-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$ and therefore $\frac{f(c)}{2} < f(x)$.] By making δ smaller if necessary, we can assume $(c - \delta, c + \delta) \subset [a, b]$. If we let $g(x)$ be the function that is equal to $\frac{f(c)}{2}$ for $c - \delta < x < c + \delta$ and is zero elsewhere, then $f(x) \geq g(x)$ for all $x \in [a, b]$, and $\int_a^b g = 2\delta \frac{f(c)}{2} > 0$. So $\int_a^b f \geq \int_a^b g > 0$.

Problem 5. Suppose that f and g are continuously differentiable functions on $[a, b]$. So, f, g, f' and g' are all continuous. Prove the *Integration by Parts* formula

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

[Hint: One way to do this is to define, for $x \in [a, b]$, $P(x) = \int_a^x f(t)g'(t)dt$ and $Q(x) = f(t)g(t)\big|_a^x - \int_a^x f'(t)g(t)dt = f(x)g(x) - f(a)g(a) - \int_a^x f'(t)g(t)dt$. Show that $P'(x) = Q'(x)$ and $P(a) = Q(a)$, and explain why this means $P(x) = Q(x)$ for all $x \in [a, b]$. Finally, use $P(b) = Q(b)$.]

Answer:

Note that since f, g, f' and g' are all continuous, it follows that fg' and $f'g$ are also continuous and hence integrable. So, the integrals in this problem are defined.

Let $P(x)$ and $Q(x)$ be as in the hint. By the Second Fundamental Theorem of Calculus, $P'(x) = f(x)g'(x)$ for all $x \in [a, b]$. Again applying the Second Fundamental Theorem and the prduct and sum rules for differentiation,

$$\begin{aligned} Q'(x) &= \frac{d}{dx}\left(f(x)g(x) - f(a)g(a) - \int_a^x f'(t)g(t)\, dx \right) \\ &= \frac{d}{dx}\big(f(x)g(x)\big) - \frac{d}{dx}\big(f(a)g(a)\big) - \frac{d}{dx}\int_a^x f'(t)g(t)\, dx \\ &= \big(f(x)g'(x) + f'(x)g(x)\big) - 0 - f'(x)g'(x) \\ &= f(x)g'(x) \\ &= P'(x) \end{aligned}$$

Since P and Q have the same derivative on $[a, b]$, they differ by a constant on that interval. Since $P(a) = Q(a) = 0$, The two functions are the same. Evaluating them at $x = b$ gives

$$\int_a^b f(t)g'(t)dt = f(t)g(t)\Big|_a^b - \int_a^b f(t)g'(t)dt$$

as we wanted to show.