This homework covers Sectsion 3.4 to 3.6. It is due on Monday, October 24.

Problem 1. We showed that if f is integrable on $[a,b]$, then $|f|$ is also integrable on $[a,b]$. Now, suppose we know that $|g|$ is integrable on $[a,b]$. Is it necessarily true that g is integrable on $[a,b]$? [Hint: Consider a simple modification of the Dirichlet function.]

Problem 2. Suppose f is a continuous function on $[a,b]$ and $f(x) > 0$ for $x \in [a,b]$. Define $F(x) = \int_a^x f$. Prove that F is strictly increasing on $[a,b]$. [Hint: This is trivial, using two facts that have already been proved.]

Problem 3 (Textbook problem 3.4.11)**.** Assume that f is integrable on $[a,b]$. Suppose that J is a real number such that $L(f,P) \leq J \leq U(f,P)$ for every partition P of $[a,b]$. Show that $J = \int_a^b f$. [Hint: Use properties of *sup* and *inf*, that is of lub and glb, and the definition of integrable.]

Problem 4. Prove the following statements.

(a) Assume that f is an integrable function on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. Prove directly, using the definition of the integral, that $\int_a^b f \geq 0$.

(b) Assume that f and g are integrable on $[a,b]$ and that $f(x) \geq g(x)$ for all $x \in [a,b]$. Prove that $\int_a^b f \geq \int_a^b g$, using part (a) and the linearity of the integral (Theorems 3.5.6 and 3.5.7).

(c) Assume that f is continuous on $[a,b]$, that $f(x) \geq 0$ for all $x \in [a,b]$, and that $f(c) > 0$, where c is some number in (a,b). Show that $\int_a^b f > 0$. [Hint: A previous homework problem already showed that there is a $\delta > 0$ such that $f(x) > \frac{f(c)}{2}$ for all $x \in (c-\delta, c+\delta)$.]

Problem 5 (Textbook problem 3.6.3)**.** Suppose that f and g are continuously differentiable functions on $[a,b]$. So, f, g, f' and g' are all continuous. Prove the *Integration by Parts* formula

$$\int_a^b f(x)'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

[Hint: One way to do this is to define, for $x \in [a,b]$, $P(x) = \int_a^x f(t)g'(t)dt$ and $Q(x) = f(t)g(t)\big|_a^x - \int_a^x f'(t)g(t)dt = f(x)g(x) - f(a)g(a) - \int_a^x f'(t)g(t)dt$. Show that $P'(x) = Q'(x)$ and $P(a) = Q(a)$, and explain why this means $P(x) = Q(x)$ for all $x \in [a,b]$.]