Problem 1. Let f be the polynomial $f(x) = 2 - 5x^2 + 3x^3 - x^4$. Use Taylor's Theorem to write f as a polynomial in powers of $(x+1)$. (That is, find the Taylor polynomial, $p_{4,-1}(x)$, of degree 4 at -1 for f.)

Answer:

Note that since $f^{(5)}(x) = 0$, the remainder term $r_{4,1}(x)$ is zero. So f is equal to its fourth degree Taylor polynomial at any point.

$$\begin{aligned} f(x) &= 2 - 5x^2 + 3x^3 - x^4 & f(-1) &= -7 \\ f'(x) &= -10x + 9x^2 - 4x^3 & f'(-1) &= 23 \\ f''(x) &= -10 + 18x - 12x^2 & f''(-1) &= -40 \\ f'''(x) &= 18 - 24x & f'''(-1) &= 42 \\ f''''(x) &= -24 & f''''(-1) &= -24 \end{aligned}$$

Then, $p_{4,-1}(x) = f(1) + f'(1)(x+1) + \frac{1}{2}f''(1)(x+1)^2 + \frac{1}{6}f'''(1)(x+1)^3 + \frac{1}{24}f''''(1)(x+1)^4$
$= -7 + 23(x+1) - 20(x+1)^2 + 7(x+1)^3 - (x+1)^4$

Problem 2. Find the general Taylor polynomial at 0, $p_{n,0}(x)$, for the function $\ln(x+1)$.

Answer:

Let $f(x) = \ln(x+1)$. We need to compute $f^{(k)}(0)$ for all $k \geq 0$. We have

$$\begin{aligned} f(x) &= \ln(x+1) & f(0) &= 0 \\ f'(x) &= \tfrac{1}{x+1} & f'(0) &= 1 \\ f''(x) &= \tfrac{-1}{(x+1)^2} & f''(0) &= -1 \\ f'''(x) &= \tfrac{2}{(x+1)^3} & f'''(0) &= 2 \\ f^{(4)}(x) &= \tfrac{-3\cdot 2}{(x+1)^4} & f^{(4)}(0) &= -3\cdot 2 \\ f^{(5)}(x) &= \tfrac{4!}{(x+1)^5} & f^{(5)}(0) &= 4! \\ f^{(6)}(x) &= \tfrac{-5!}{(x+1)^6} & f^{(6)}(0) &= -5! \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \tfrac{(-1)^{n+1}(n-1)!}{(x+1)^n} & f^{(k)}(0) &= (-1)^{n+1}(n-1)! \end{aligned}$$

We see that $\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k+1}(k-1)!}{k!} = \frac{(-1)^{k+1}}{k}$. So for the n^{th} Taylor polynomial at 0, we get

$$p_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k}$$

Problem 3 (from 4.2.14 from the textbook). We have shown that the n^{th} Taylor polynomial for e^x at 0 is $p_{n,0}(x) = \sum_{k=1}^{n} \frac{1}{n!}x^n$. Show that e is irrational by using proof by contradiction. Suppose, for the sake of contradiction, that $e = \frac{p}{q}$ for some integers p and q.

(a) Use the Lagrange form of the remainder term from Taylor's Theorem to show that there is a $c \in [0,1]$ such that $\frac{p}{q} - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = \frac{e^c}{(n+1)!}$.

(b) Multiply both sides of the equation in (a) by $n!$, and show that left side of the resulting equation is an integer when $n \geq q$.

(c) Show that the right side of the equation that you got in part (b) is not an integer when $n > e$. Conclude that e is irrational.

Answer:

(a) Suppose, for the sake of contradiction, that $e = \frac{p}{q}$, where p and q are integers. Since the n^{th} Taylor polynomial for e^x at 0 is $p_{n,0}(x) = \sum_{k=1}^{n} \frac{1}{k!}x^k$, and we know that $\frac{p}{q} = e^1 = p_{n,0}(1) + r_{n,0}(1)$, we see that $\frac{p}{q} - \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)$ is the remainder term, $r_{n,0}(1)$. Using the Lagrange form of the remainder ($r_{n,0}(x) = \frac{f^{n+1}(c)}{(n+1)!}x^{n+1}$), we get that there is a $c \in [0,1]$ such that $r_{n,0}(1) = \frac{f^{n+1}(c)}{(n+1)!} = \frac{e^c}{(n+1)!}$.

(b) Multiplying both sides of the equation by $n!$ gives $p \cdot \frac{n!}{q} - \left(\frac{n!}{0!} + \frac{n!}{1!} + \cdots + \frac{n!}{n!}\right) = \frac{e^c}{n+1}$. If $n \geq q$, then q is one of the factors in the product $n! = 1 \cdot 2 \cdot 3 \cdots n$, so $\frac{n!}{q}$ is also an integer. All the other terms on the left side are integers, so the left side of the equation is an integer when $n \geq q$.

(c) Note that since $0 \leq c \leq 1$ and e^x is an increasing function, we have $e^c \leq e^1 = e$. If $n > e$, then the fraction $\frac{e^c}{n+1}$ is strictly between 0 and 1 and so is not an integer. For any $n > \max(q, e)$, the non-integer on the right side of the equation cannot equal the integer on the left side. This contradiction shows that e cannot be rational.

Problem 4. Suppose that f is a function defined for all $x \geq 1$ and that $\lim_{x \to +\infty} f(x) = L$. Define a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = f(n)$ for all $n \in \mathbb{N}$. Prove that $\lim_{n \to \infty} a_n = L$.

Answer:

Let $\varepsilon > 0$. We must find $N \in \mathbb{N}$ such that for $n \in N$, $n \geq N$ implies $|a_n - L| < \varepsilon$. Since $\lim_{x \to +\infty} f(x) = L$, there is an $M \in \mathbb{R}$ such that for $x \in \mathbb{R}$, $x > M$ implies $|f(x) - L| < \varepsilon$. Let N be any integer greater than M, then for an integer $n > N$, $|a_n - L| = |f(n) - L| < \varepsilon$.

Problem 5. Prove that if $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence that is not bounded above, then $\lim_{n \to \infty} x_n = +\infty$.

Answer:

Let $M \in \mathbb{R}$. We want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$. Since the sequence is not bounded above, M cannot be an upper bound for $\{x_1, x_2, \ldots\}$. So, there is an an $N \in \mathbb{N}$ such that $x_N > M$. But for any $n > N$, we know $x_n \geq x_N$ since $\{x_n\}_{n=1}^{\infty}$ is increasing. So let $n > N$. Then we have $x_n \geq x_N > M$. That is, $x_n > M$ for any $n > N$, as we wanted to show.

Problem 6 (From Textbook problem 4.2.5)**.** Let $\{a_n\}_{n=1}^{\infty}$ be defined inductively as follows:

$$a_1 = 1, \qquad a_n = 1 + \frac{a_{n-1}}{4} \quad \text{for } n > 1$$

(a) Show by induction that a_n is bounded above by 4/3.

(b) Show that $\{a_n\}_{n=1}^{\infty}$ is convergent by showing that it is increasing.

(c) Show that $\lim_{n\to\infty} a_n = 4/3$. [Hint: Use the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ and the recursive definition of a_n.]

Answer:

(a) $a_n = 1$, so $a_n < 4/3$ for $n = 1$. Suppose that we know $a_k < 4/3$ for some $k \geq 1$. To complete the induction, we must show that $a_{k+1} < 4/3$. But $a_{k+1} = 1 + \frac{a_k}{4} < 1 + \frac{4/3}{4} = 1 + 1/3 = 4/3$.

(b) Since $a_n < 4/3$, we see that $a_{n+1} - a_n = \left(1 + \frac{a_n}{4}\right) - a_n = 1 - \frac{3}{4}a_n > 1 - \frac{3}{4}\cdot\frac{4}{3} = 0$. So $a_{n+1} > a_n$, which means $\{a_n\}_{n=1}^{\infty}$ is increasing. Since the sequence is increasing and bounded above, it is convergent.

(c) Let $z = \lim_{n\to\infty} a_n$. Then $z = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1 + \frac{a_n}{4}\right) = 1 + \frac{1}{4}\cdot\lim_{n\to\infty} a_n = 1 + \frac{1}{4}z$. Solving for z, we get $z - \frac{1}{4}z = 1$, or $\frac{3}{4}z = 1$, or $z = \frac{4}{3}$.

Problem 7. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences, and $\{a_n\}_{n=1}^{\infty}$ is convergent with $\lim_{n\to\infty} a_n = L$. Suppose in addition that $\lim_{n\to\infty} |a_n - b_n| = 0$. Show that $\{b_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} b_n = L$.

Answer:

To show $\lim_{n\to\infty} b_n = L$, let $\varepsilon > 0$. We must find an $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - L| < \varepsilon$. Since $\lim_{n\to\infty} a_n = L$, there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - L| < \frac{\varepsilon}{2}$. Since $\lim_{n\to\infty} |a_n - b_n| = 0$, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - b_n| < \frac{\varepsilon}{2}$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have $|b_n - L| = |(b_n - a_n) + (a_n - L)| \leq |b_n - a_n| + |a_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.