Problem 1. State a careful definition of $\lim_{x \to a^+} f(x) = +\infty$. Then use the definition to prove directly that $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.

Answer:

Define $\lim_{x \to a^+} f(x) = +\infty$ if for every $M \in \mathbb{R}$, there is a $\delta > 0$ such that for all x, if $0 < x - a < \delta$, then $f(x)$ is defined and $f(x) > M$.

To show that $\lim_{x \to 0^+} \frac{1}{x} = +\infty$, let $M \in \mathbb{R}$. In case $M \le 0$, δ can be arbitrary because $\frac{1}{x} > 0$ for all $x > 0$. So, consider the case where $M > 0$. Let $\delta = \frac{1}{M}$, and suppose that x satisfies $0 < x - 0 < \delta$. We must show that $\frac{1}{x} > M$. But $0 < x - 0 < \delta$ means x is positive and $x < \frac{1}{M}$. Since M and x are positive, multiplying the inequality by M and dividing by x gives $M < \frac{1}{x}$, as we wanted to show.

Problem 2. Let X and Y be non-empty, bounded subsets of $\mathbb{R}$. Suppose that for every $x \in X$ and for every $y \in Y$, $x < y$. Prove that $lub(X) \le glb(Y)$. Is it always true that $lub(X) < glb(Y)$? (Prove or give a counterexample!)

Answer:

Consider any $x \in X$. Since $x < y$ for all $y \in Y$, x is a lower bound for y. By defnition of greatest lower bound, this implies that $x < glb(Y)$. Since that is true for all $x \in X$, $glb(Y)$ is an upper bound for X. By definition of least upper bound, this implies that $lub(X) < glb(Y)$,

It is not always the case that $lub(X) < glb(Y)$. For a counterexample, let $X = [0, 1)$ and let $Y = (1, 2]$. Then $x < y$ for all $x \in X$ and $y \in Y$, but $lub(X) = glb(Y)$,

(Alternative proof: Let $\varepsilon > 0$. We know that there is some $x \in X$ such that $x > lub(X) - \varepsilon$, and there is some $y \in Y$ such that $y < glb(Y) + \varepsilon$. Since $x \in X$ and $y \in Y$, we know by assumption that $x < y$. So we have $lub(X) - \varepsilon < x < y < glb(Y) + \varepsilon$, and therefore $lub(X) < glb(Y) + 2\varepsilon$. Since this is true for any $\varepsilon > 0$, $lub(X) \le glb(Y)$.)

Problem 3. Let A and B be subsets of $\mathbb{R}$. Suppose that x is an accumulation point of the set $A \cup B$. Show that x is an accumulation point of A or x is an accumulation point of B (or both). (Hint: Try a proof by contradiction.)

Answer:

Suppose, for the sake of contradiction, that x is not an accumulation point of A and x is not an accumulation point of B. Since x is not an accumulation point of A, there is an $\eta > 0$ such that $A \cap (x - \eta, x + \eta)$ contains no point of A other than, possibly, x. Since x is not an accumulation point of B, there is a $\zeta > 0$ such that $A \cap (x - \zeta, x + \zeta)$ contains no point of A other than, possibly, x. Let $\varepsilon = \min(\eta, \zeta)$. Then $(x - \varepsilon, x + \varepsilon)$ contains no point of A other than x, and it also contains no point of B other than x, That is, $(x - \varepsilon, x + \varepsilon)$ contains no point of $A \cup B$ other than x. By definition of accumulation point, this means that x is not an accumulation point of $A \cup B$. But that contradicts the hypothesis.

Problem 4. Let f and g be functions. Then we can define a new function $\max(f, g)$ whose value at x is given by $\max(f(x), g(x))$.

(a) Show that for any numbers a and b, $\max(a, b) = \frac{1}{2}\big(|a - b| + a + b\big)$. (Hint: Consider two cases.)

(b) Now, suppose that f and g are continuous on an interval I. Show that the function $\max(f, g)$ is also continuous on I. Be clear about what continuity rules or theorems you use.

Answer:

(a) Consider the cases $a < b$ and $a \geq b$. In the case $a < b$, $\max(a, b) = b$. We have $a - b < 0$, and therefore $|a - b| = b - a$. So in this case, $\frac{1}{2}\big(|a-b|+a+b\big) = \frac{1}{2}\big(b-a+a+b\big) = \frac{1}{2}(2b) = b = \max(a, b)$. And in the case $a \geq b$, $\max(a, b) = a$. We have $a - b > 0$, and therefore $|a-b| = a-b$. So in this case, $\frac{1}{2}\big(|a-b|+a+b\big) = \frac{1}{2}\big(a-b+a+b\big) = \frac{1}{2}(2a) = a = \max(a, b)$.

(b) By part (a), the function $\max(f, g)$ is given by $\frac{1}{2}\big(|f - g| + f + b\big)$. We know the difference of two continuous functions is continuous, the absolute value function is continuous, and the composition of continuous functions is continuous. So, $|f - g|$ is a continuous function. Then, since the sum of continuous functions is continuous, we know $|f - g| + f + g$ is continuous. Finally, since a constant multiple of a continuous function is continuous, we get that $\frac{1}{2}\big(|f - g| + f + b\big)$. That is, $\max(f, g)$ is continuous.

Problem 5. Let S be a subset of $\mathbb{R}$. Recall that S is said to be *dense* in $\mathbb{R}$ if for any open interval (a, b), the intersection of S with the set (a, b) is not empty. (That is, there is at least one $s \in S$ such that $a < s < b$.) Prove that S is dense in $\mathbb{R}$ if and only if every point of $\mathbb{R}$ is an accumulation point of S.

Answer:

$\Longrightarrow$) Suppose that S is a dense subset of $\mathbb{R}$. Let $x \in \mathbb{R}$. We must show that x is an accumulation point of S. Let $\varepsilon > 0$. We want to find $s \in S$ such that $0 < |x - s| < \varepsilon$. Since S is dense, there is some $s \in S$ such that s is in the open interval $(x, x + \varepsilon)$. So, $s \neq x$ (giving $0 < |x - s)$, and $x < s < x + \varepsilon$ (giving $|x - s| < \varepsilon$).

$\Longleftarrow$) Suppose that every point of $\mathbb{R}$ is an accumulation point of S. We must show S is dense in $\mathbb{R}$. Let $a, b \in \mathbb{R}$ with $a < b$. We must find some $s \in S$ such that $a < s < b$. Let $x = \frac{b+a}{2}$, the midpoint of (a, b), and let $\varepsilon = \frac{b-a}{2}$, half the length of (a, b). Since x is an accumulation point of S, there is some $s \in S$ such that $0 < |x - s| < \varepsilon$. So $\left|s - \frac{b+a}{2}\right| < \frac{b-a}{2}$. This is equivalent to

$$
\begin{aligned}
-\frac{b-a}{2} < s - \frac{b+a}{2} < \frac{b-a}{2} \\
\frac{b+a}{2} - \frac{b-a}{2} < s < \frac{b+a}{2} + \frac{b-a}{2} \\
\frac{b+a-b+a}{2} < s < \frac{b+a+b-a}{2} \\
a < s < b
\end{aligned}
$$

which is what we needed to show.

Problem 6. Let $f(x)$ be a continuous function on a closed, bounded interval $[a, b]$. In class, we used uniform continuity of f to show that f is bounded above. However, it is possible to prove that directly using the Heine-Borel Theorem. Follow this outline to prove that there is a number M such that $f(x) \leq M$ for all $x \in [a, b]$:

- Show that for any $z \in [a, b]$, there is a $\delta_z > 0$ and a number M_z such that $f(x) \leq M_z$ for all $x \in (z - \delta_z, z + \delta_z)$. (This is an easy consequence of continuity. Just let $\varepsilon = 1$ in the definition of continuity at z, and get $f(x) < f(z) + 1$ for all x near enough to z.)
- Define an open cover of $[a, b]$ consisting of the intervals $(z - \delta_z, z + \delta_z)$, for all $z \in [a, b]$. (State why it is a cover.)
- Apply the Heine-Borel Theroem, and finish the proof.

Answer:

Suppose f is continuous on $[a, b]$. Let $z \in [a, b]$. By definition of continuity at z, letting ε in that definition equal 1, there is a $\delta_z > 0$ such that for all $x \in [a, b]$, if $|x - z| < \delta_z$, then $|f(x) - f(z)| < 1$. Now, $|f(x) - f(z)| < 1$ is equivalent to $-1 < f(x) - f(z) < 1$, or $f(z) - 1 < f(x) < f(z) + 1$. Note in particular that $f(x) < f(z) + 1$ for all $x \in (z - \delta_x, z + \delta_z)$. Let $M_z = f(x) + 1$.

The set $\mathscr{C} = \{(z - \delta_x, z + \delta_z) : z \in [a, b]\}$ is an open cover of $[a, b]$ since every $c \in [a, b]$ is in the open set $(c - \delta_c, c + \delta_c)$, which is one of the sets in $\mathscr{C}$.

By the Heine-Borel Theorem, there is a finite subcover of $[a, b]$ from $\mathscr{C}$. Let that subcover be $\mathscr{D} = \{(z_i - \delta_{z_i}, z_i + \delta_{z_i}) : i = 1, 2, \dots, k\}$, and let $M = \max(M_{z_1}, M_{z_2}, \dots, M_{z_k})$. We must show that $f(x) \leq M$ for all $x \in [a, b]$. Let $x \in [a, b]$. Since $\mathscr{D}$ covers $[a, b]$, $x \in (z_i - \delta_{z_i}, z_i + \delta_{z_i})$ for some i, so we have $f(x) < M_{z_i} \leq M$.

Problem 7. Suppose that $f(x)$ and $g(x)$ are uniformly continuous on the interval I (which is not necessarily closed or bounded). Show directly from the definition of uniform continuity that $f(x) + g(x)$ is uniformly continuous on I.

Answer:

Suppose f and g are uniformly continuous on an interval I. We want to show that $f + g$ is uniformly continuous on I. Let $\varepsilon > 0$.

Since f is uniformly continuous on I, there is a $\delta_1 > 0$ such that for every $x, y \in I$, if $|x - y| < \delta_1$, then $|f(x) - f(x)| < \frac{\varepsilon}{2}$.

Since g is uniformly continuous on I, there is a $\delta_2 > 0$ such that for every $x, y \in I$, if $|x - y| < \delta_2$, then $|g(x) - g(x)| < \frac{\varepsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. Let $x, y \in I$ such that $|x - y| < \delta_1$. We then have both $|f(x) - f(y)| < \frac{\varepsilon}{2}$ and $|g(x) - g(y)| < \frac{\varepsilon}{2}$. So

$$\begin{aligned} \big|(f(x) + g(x)) - (f(y) + g(y))\big| &= \big|(f(x) - f(y)) + (g(x) - g(y))\big| \\ &\leq (|f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

This shows that $f + g$ is uniformly continuous on I.