Problem 1. State a careful definition of \( \lim_{x \to a^+} f(x) = +\infty \). Then use the definition to prove directly that \( \lim_{x \to 0^+} \frac{1}{x} = +\infty \).

Answer:

Define \( \lim_{x \to a^+} f(x) = +\infty \) if for every \( M \in \mathbb{R} \), there is a \( \delta > 0 \) such that for all \( x \), if \( 0 < x - a < \delta \), then \( f(x) > M \).

To show that \( \lim_{x \to 0^+} \frac{1}{x} = +\infty \), let \( M \in \mathbb{R} \). In case \( M \leq 0 \), \( \delta \) can be arbitrary because \( \frac{1}{x} > 0 \) for all \( x > 0 \). So, consider the case where \( M > 0 \). Let \( \delta = \frac{1}{M} \), and suppose that \( x \) satisfies \( 0 < x - 0 < \delta \). We must show that \( \frac{1}{x} > M \). But \( 0 < x - 0 < \delta \) means \( x \) is positive and \( x < \frac{1}{M} \). Since \( M \) and \( x \) are positive, multiplying the inequality by \( M \) and dividing by \( x \) gives \( M < \frac{1}{x} \), as we wanted to show.

Problem 2. Let \( X \) and \( Y \) be non-empty, bounded subsets of \( \mathbb{R} \). Suppose that for every \( x \in X \) and for every \( y \in Y \), \( x < y \). Prove that \( \text{lub}(X) \leq \text{glb}(Y) \). Is it always true that \( \text{lub}(X) < \text{glb}(Y) \)? (Prove or give a counterexample!)

Answer:

Consider any \( x \in X \). Since \( x < y \) for all \( y \in Y \), \( x \) is a lower bound for \( Y \). By definition of greatest lower bound, this implies that \( x < \text{glb}(Y) \). Since that is true for all \( x \in X \), \( \text{glb}(Y) \) is an upper bound for \( X \). By definition of least upper bound, this implies that \( \text{lub}(X) < \text{glb}(Y) \).

It is not always the case that \( \text{lub}(X) < \text{glb}(Y) \). For a counterexample, let \( X = [0, 1] \) and let \( Y = (1, 2] \). Then \( x < y \) for all \( x \in X \) and \( y \in Y \), but \( \text{lub}(X) = \text{glb}(Y) \).

(Alternative proof: Let \( \varepsilon > 0 \). We know that there is some \( x \in X \) such that \( x > \text{lub}(X) - \varepsilon \), and there is some \( y \in Y \) such that \( y < \text{glb}(Y) + \varepsilon \). Since \( x \in X \) and \( y \in Y \), we know by assumption that \( x < y \). So we have \( \text{lub}(X) - \varepsilon < x < y < \text{glb}(Y) + \varepsilon \), and therefore \( \text{lub}(X) < \text{glb}(Y) + 2\varepsilon \). Since this is true for any \( \varepsilon > 0 \), \( \text{lub}(X) \leq \text{glb}(Y) \).)

Problem 3. Let \( A \) and \( B \) be subsets of \( \mathbb{R} \). Suppose that \( x \) is an accumulation point of the set \( A \cup B \). Show that \( x \) is an accumulation point of \( A \) or \( x \) is an accumulation point of \( B \) (or both). (Hint: Try a proof by contradiction.)

Answer:

Suppose, for the sake of contradiction, that \( x \) is not an accumulation point of \( A \) and \( x \) is not an accumulation point of \( B \). Since \( x \) is not an accumulation point of \( A \), there is a \( \eta > 0 \) such that \( A \cap (x - \eta, x + \eta) \) contains no point of \( A \) other than, possibly, \( x \). Since \( x \) is not an accumulation point of \( B \), there is a \( \zeta > 0 \) such that \( A \cap (x - \zeta, x + \zeta) \) contains no point of \( A \) other than, possibly, \( x \). Let \( \varepsilon = \min(\eta, \zeta) \). Then \( (x - \varepsilon, x + \varepsilon) \) contains no point of \( A \) other than \( x \), and it also contains no point of \( B \) other than \( x \). That is, \( (x - \varepsilon, x + \varepsilon) \) contains no point of \( A \cup B \) other than \( x \). By definition of accumulation point, this means that \( x \) is not an accumulation point of \( A \cup B \). But that contradicts the hypothesis.
**Problem 4.** Let \( f \) and \( g \) be functions. Then we can define a new function \( \max(f, g) \) whose value at \( x \) is given by \( \max(f(x), g(x)) \).

(a) Show that for any numbers \( a \) and \( b \), \( \max(a, b) = \frac{1}{2}(|a-b| + a + b) \). (Hint: Consider two cases.)

(b) Now, suppose that \( f \) and \( g \) are continuous on an interval \( I \). Show that the function \( \max(f, g) \) is also continuous on \( I \). Be clear about what continuity rules or theorems you use.

**Answer:**

(a) Consider the cases \( a < b \) and \( a \geq b \). In the case \( a < b \), \( \max(a, b) = b \). We have \( a-b < 0 \), and therefore \( |a-b| = b-a \). So in this case, \( \frac{1}{2}(|a-b| + a + b) = \frac{1}{2}(b-a+a+b) = \frac{1}{2}(2b) = b = \max(a, b) \). And in the case \( a \geq b \), \( \max(a, b) = a \). We have \( a-b > 0 \), and therefore \( |a-b| = a-b \). So in this case, \( \frac{1}{2}(|a-b| + a + b) = \frac{1}{2}(a-b+a+b) = \frac{1}{2}(2a) = a = \max(a, b) \).

(b) By part (a), the function \( \max(f, g) \) is given by \( \frac{1}{2}(|f-g| + f + b) \). We know the difference of two continuous functions is continuous, the absolute value function is continuous, and the composition of continuous functions is continuous. So, \( |f-g| \) is a continuous function. Then, since the sum of continuous functions is continuous, we know \( |f-g| + f + g \) is continuous. Finally, since a constant multiple of a continuous function is continuous, we get that \( \frac{1}{2}(|f-g| + f + g) \). That is, \( \max(f, g) \) is continuous.

**Problem 5.** Let \( S \) be a subset of \( \mathbb{R} \). Recall that \( S \) is said to be dense in \( \mathbb{R} \) if for any open interval \((a, b)\), the intersection of \( S \) with the set \((a, b)\) is not empty. (That is, there is at least one \( s \in S \) such that \( a < s < b \).) Prove that \( S \) is dense in \( \mathbb{R} \) if and only if every point of \( \mathbb{R} \) is an accumulation point of \( S \).

**Answer:**

\[ \implies \] Suppose that \( S \) is a dense subset of \( \mathbb{R} \). Let \( x \in \mathbb{R} \). We must show that \( x \) is an accumulation point of \( S \). Let \( \varepsilon > 0 \). We want to find \( s \in S \) such that \( 0 < |x-s| < \varepsilon \). Since \( S \) is dense, there is some \( s \in S \) such that \( s \) is in the open interval \((x, x+\varepsilon)\). So, \( s \neq x \) (giving \( 0 < |x-s| \)), and \( x < s < x+\varepsilon \) (giving \( |x-s| < \varepsilon \)).

\[ \iff \] Suppose that every point of \( \mathbb{R} \) is an accumulation point of \( S \). We must show \( S \) is dense in \( \mathbb{R} \). Let \( a, b \in \mathbb{R} \) with \( a < b \). We must find some \( s \in S \) such that \( a < s < b \). Let \( x = \frac{b+a}{2} \), the midpoint of \((a, b)\), and let \( \varepsilon = \frac{b-a}{2} \), half the length of \((a, b)\). Since \( x \) is an accumulation point of \( S \), there is some \( s \in S \) such that \( 0 < |x-s| < \varepsilon \). So \( |s - \frac{b+a}{2}| < \frac{b-a}{2} \). This is equivalent to

\[
\begin{align*}
\frac{-b-a}{2} < s - \frac{b+a}{2} < \frac{b-a}{2} \\
\frac{b+a}{2} - \frac{b-a}{2} < s < \frac{b+a}{2} + \frac{b-a}{2} \\
\frac{b+a-a+b}{2} < s < \frac{b+a+b-a}{2} \\
a < s < b
\end{align*}
\]

which is what we needed to show.
Problem 6. Let \( f(x) \) be a continuous function on a closed, bounded interval \([a, b]\). In class, we used uniform continuity of \( f \) to show that \( f \) is bounded above. However, it is possible to prove that directly using the Heine-Borel Theorem. Follow this outline to prove that there is a number \( M \) such that \( f(x) \leq M \) for all \( x \in [a, b] \):

- Show that for any \( z \in [a, b] \), there is a \( \delta_z > 0 \) and a number \( M_z \) such that \( f(x) \leq M_z \) for all \( x \in (z - \delta_z, z + \delta_z) \). (This is an easy consequence of continuity. Just let \( \varepsilon = 1 \) in the definition of continuity at \( z \), and get \( f(x) < f(z) + 1 \) for all \( x \) near enough to \( z \).)

- Define an open cover of \([a, b]\) consisting of the intervals \((z - \delta_z, z + \delta_z)\), for all \( z \in [a, b] \).
  (State why it is a cover.)

- Apply the Heine-Borel Theorem, and finish the proof.

Answer:

Suppose \( f \) is continuous on \([a, b]\). Let \( z \in [a, b] \). By definition of continuity at \( z \), letting \( \varepsilon \) in that definition equal 1, there is a \( \delta_z > 0 \) such that for all \( x \in [a, b] \), if \( |x - z| < \delta_z \), then \( |f(x) - f(z)| < 1 \). Now, \( |f(x) - f(z)| < 1 \) is equivalent to \(-1 < f(x) - f(z) < 1\), or \( f(z) - 1 < f(x) < f(z) + 1 \). Note in particular that \( f(x) < f(z) + 1 \) for all \( x \in (z - \delta_z, z + \delta_z) \).

Let \( M_z = f(x) + 1 \).

The set \( C = \{(z - \delta_z, z + \delta_z) : z \in [a, b]\} \) is an open cover of \([a, b]\) since every \( c \in [a, b] \) is in the open set \((c - \delta_c, c + \delta_c)\), which is one of the sets in \( C \).

By the Heine-Borel Theorem, there is a finite subcover of \([a, b]\) from \( C \). Let that subcover be \( D = \{(z_i - \delta_{z_i}, z_i + \delta_{z_i}) : i = 1, 2, \ldots, k\} \), and let \( M = \max(M_{z_1}, M_{z_2}, \ldots, M_{z_k}) \). We must show that \( f(x) \leq M \) for all \( x \in [a, b] \). Let \( x \in [a, b] \). Since \( D \) covers \([a, b]\), \( x \in (z_i - \delta_{z_i}, z_i + \delta_{z_i}) \) for some \( i \), so we have \( f(x) < M_z \leq M \).

Problem 7. Suppose that \( f(x) \) and \( g(x) \) are uniformly continuous on the interval \( I \) (which is not necessarily closed or bounded). Show directly from the definition of uniform continuity that \( f(x) + g(x) \) is uniformly continuous on \( I \).

Answer:

Suppose \( f \) and \( g \) are uniformly continuous on an interval \( I \). We want to show that \( f + g \) is uniformly continuous on \( I \). Let \( \varepsilon > 0 \).

Since \( f \) is uniformly continuous on \( I \), there is a \( \delta_1 > 0 \) such that for every \( x, y \in I \), if \( |x - y| < \delta_1 \), then \( |f(x) - f(y)| < \frac{\varepsilon}{2} \).

Since \( g \) is uniformly continuous on \( I \), there is a \( \delta_2 > 0 \) such that for every \( x, y \in I \), if \( |x - y| < \delta_2 \), then \( |g(x) - g(y)| < \frac{\varepsilon}{2} \).

Let \( \delta = \min(\delta_1, \delta_2) \). Let \( x, y \in I \) such that \( |x - y| < \delta \). We then have both \( |f(x) - f(y)| < \frac{\varepsilon}{2} \) and \( |g(x) - g(y)| < \frac{\varepsilon}{2} \). So

\[
|f(x) + g(x) - (f(y) + g(y))| = |(f(x) - f(y)) + (g(x) - g(y))| \\
\leq |f(x) - f(y)| + |g(x) - g(y)| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon
\]

This shows that \( f + g \) is uniformly continuous on \( I \).