Problem 1. (a) Suppose that the function $F(x)$ is differentiable at a. Show directly from the definition of derivative that the function $G(x) = F(x)^2$ is differentiable at a and $G'(a) = 2F(a)F'(a)$. [Hint: You only need to factor $F(x)^2 - F(a)^2$ in the definition.]

(b) We know that $f(x)g(x) = \frac{1}{4}\big((f(x)+g(x))^2 - (f(x)-g(x))^2\big)$ from a previous homework problem . Using only this fact, the result from part (a), and the sum, difference, and constant multiple rules for derivatives, find the formula for the derivative of $f(x)g(x)$,

Answer:

(a) Since F is differentiable at a, $\lim_{x\to a} \frac{F(x)-F(a)}{x-a} = F'(a)$. Furthermore, differentiability implies continuity, so F it is continuous at a, meaning $\lim_{x\to a} F(x) = F(a)$. Therefore,

$$\begin{aligned}
G'(a) &= \lim_{x\to a} \frac{F(x)^2 - F(a)^2}{x-a} \\
&= \lim_{x\to a} \frac{\big(F(x)+F(a)\big)\big(F(x)-F(a)\big)}{x-a} \\
&= \lim_{x\to a} \big(F(x)+F(a)\big)\frac{F(x)-F(a)}{x-a} \\
&= \big(\lim_{x\to a} F(x) + \lim_{x\to a} F(a)\big) \cdot \lim_{x\to a} \frac{F(x)-F(a)}{x-a} \\
&= \big(F(a)+F(a)\big)\cdot F'(a) \\
&= 2F(a)F'(a)
\end{aligned}$$

(b) We can then compute

$$\begin{aligned}
\big(f(x)g(x)\big)' &= \big(\frac{1}{4}((f(x)+g(x))^2 - (f(x)-g(x))^2)\big)' \\
&= \frac{1}{4}\Big((f(x)+g(x))^2 - (f(x)-g(x))^2\Big)' \\
&= \frac{1}{4}\Big(\big((f(x)+g(x))^2\big)' - \big((f(x)-g(x))^2\big)'\Big) \\
&= \frac{1}{4}\Big(\big(2(f(x)+g(x))(f(x)+g(x))'\big) - \big(2(f(x)-g(x))(f(x)-g(x))'\big)\Big) \\
&= \frac{1}{4}\Big(\big(2(f(x)+g(x))(f'(x)+g'(x))\big) - \big(2(f(x)-g(x))(f'(x)-g'(x))\big)\Big) \\
&= \frac{1}{4}\Big(\big(2f(x)f'(x) + 2f(x)g'(x) + 2g(x)f'(x) + 2g(x)g'(x)\big) - \\
&\qquad\qquad \big(2f(x)f'(x) - 2f(x)g'(x) - 2g(x)f'(x) + 2g(x)g'(x)\big)\Big) \\
&= \frac{1}{4}\big(4f(x)g'(x) + 4g(x)f'(x)\big) \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

Problem 2. Let f and g be differentiable functions on $[a,b]$. Suppose that $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x \in (a,b)$. Prove that $f(b) > g(b)$. [Hint: Consider the function $h(x) = f(x) - g(x)$ and apply the Mean Value Theorem.]

Answer:

Let $h(x) = f(x) - g(x)$. Note that $h(a) = f(a) - g(a) = 0$, and $h'(x) = f'(x) - g'(x) > 0$ for all $x \in (a,b)$. h is differentiable on $[a,b]$, so the Mean Value Theorem applies to h. That is, there is a $c \in (a,b)$ such that $h'(c) = \frac{h(b)-h(a)}{b-a}$. We know $b - a > 0$ and $h'(c) > 0$, so $h(b) - h(a) = (b-a)f'(c) > 0$. Since $h(a) = 0$, we get that $h(b) > 0$. That is, $f(b) - g(b) > 0$, and $f(b) > g(b)$.

Problem 3. Let f be an integrable function on $[a,b]$. Suppose that $A \leq f(x) \leq B$ for all $x \in [a,b]$. Show, from the definition of the integral, that $A \cdot (b-a) \leq \int_a^b f \leq B \cdot (b-a)$. (Hint: Use the trivial partition $P = \{x_0, x_1\}$ where $x_0 = a$, $x_1 = b$.)

Answer:

We know that for any partition P of $[a,b]$, $L(f,P) \leq \int_a^b \leq U(f,P)$. Consider the trivial partition $P = \{x_0, x_1\}$ where $x_0 = a$, $x_1 = b$. Then $L(f,P) = m \cdot (x_1 - x_0) = m \cdot (b-a)$, where $m = \inf\{f(x) \mid x \in [a,b]\}$, and $U(f,P) = M \cdot (x_1 - x_0) = M \cdot (b-a)$, where $M = \sup\{f(x) \mid x \in [a,b]\}$.

Saying $A \leq f(x)$ for all $x \in [a,b]$ means that A is a lower bound for $\{f(x) \mid x \in [a,b]\}$. So $A \leq \inf\{f(x) \mid x \in [a,b]\}$. That is $A \leq m$. Similarly, B is an upper bound for $\{f(x) \mid x \in [a,b]\}$, and $B \geq M$. So we have

$$A \cdot (b-a) \leq m \cdot (b-a) = L(f,P) \leq \int_a^b f \leq U(f,P) = M \cdot (b-a) \leq B \cdot (b-a)$$

Problem 4. Suppose that f is integrable on $[a,b]$. Define $F(x) = \int_a^x f$ for $x \in [a,b]$, and define $G(x) = \int_a^x F$ for $x \in [a,b]$. How do we know $\int_a^x F$ exists? Show that G is differentiable on $[a,b]$.

Answer:

We know that F(x) is continuous [Theorem 3.6.1] and therefore integrable [Theorem 3.5.1] on $[a,x]$ for any $x \in (a,b]$. That is, $\int_a^x F$ exists.

Furthermore, since F is a continuous function on $[a,b]$, and $G(x) = \int_a^x F$ for $x \in [a,b]$, we know by the Second Fundamental Theorem of Calculus that G is differentiable on $[a,b]$ (and that $G'(x) = F(x)$).

Problem 5. Let $\sum_{k=1}^{\infty} a_k$ be a convergent series of non-negative terms. Prove that the series $\sum_{k=1}^{\infty} a_k^2$ also converges. [Hints: $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and remember that you only need to show $\sum_{k=N}^{\infty} a_k^2$ converges for some N.]

Answer:

Since $\sum_{k=1}^{\infty} a_k$ converges, we know that $\lim_{n\to\infty} a_k = 0$. By the definition of limit of a sequence (taking $\varepsilon = 1$ in that definition), there is an $N \in \mathbb{N}$ such that for any $k \geq N$, $|a_k - 0| < 1$. Now, a_k is non-negative, so we have $0 \leq a_k < 1$ for all $k \geq N$. Furthermore, $0 \leq a_k < 1$ implies $a_k^2 \leq a_k$. We know that $\sum_{k=N}^{\infty} a_k$ converges because $\sum_{k=1}^{\infty} a_k$ converges. By the comparison test, since $a_k^2 \leq a_k$ for $k \geq N$, we get that $\sum_{k=N}^{\infty} a_k^2$ converges. Finally, that implies that $\sum_{k=1}^{\infty} a_k^2$ converges

Problem 6 (*Textbook problem 4.5.7, 8*). **(a)** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval I. Assume that each f_n is bounded; that is, there are constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in I$. Prove: If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then f must also be bounded on I.

(b) Show that the hypothesis of uniform convergence is necessary by finding a sequence of bounded functions that converges pointwise to a function that is not bounded. ([Hint: Take $I = [0, \infty)$ and look for a simple example.]

Answer:

(a) We know from the definition of uniform convergence (taking $\varepsilon = 1$ in that definition), that there is an $n \in N$ such that for all $n \geq N$ and all $x \in I$, $|f_n(x) - f(x)| < 1$. In particular, $|f(x) - f_N(x)| < 1$. We know by assumption that $|f_N(x)| \leq M_N$ for all $x \in I$. So we get for all $x \in I$,

$$\begin{aligned} |f(x)| &= |f(x) - f_N(x) + f_N(x)| \\ &\leq |f(x) - f_N(x)| + |f_N(x)| \\ &< 1 + M_N \end{aligned}$$

That is, we have shown that $1 + M_N$ is a bound for f on I.

(b) Define a sequence of functions on the interval $I = [0, \infty)$ by $f_n(x) = \begin{cases} x & \text{if } x < n \\ n & \text{if } x \geq n \end{cases}$. Then $f_n(x) \leq n$ for all $x \geq 0$, so f_n is bounded by n on I. It is clear that $\lim_{n\to\infty} f_n(x) = x$ for all $x \in I$, because in fact $f_n(x) = x$ for all $n > x$. So the pointwise limit of $\{f_n\}$ is the function $f(x) = x$, which is not bounded on $[0, \infty)$.

Problem 7. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq r|x - y|$ for all $x, y \in \mathbb{R}$, where r is a constant in the interval $0 \leq r < 1$. Such a function is said to be a **contraction** on $\mathbb{R}$. Note that a contraction is simply a Lipschitz function with Lipschitz constant strictly less than 1, so we already know that f is continuous.

(a) Let t be any real number. Define a sequence $\{a_n\}_{n=0}^{\infty}$ by $a_0 = t$, $a_n = f(a_{n-1})$ for $n > 0$. That is $a_0 = t, a_1 = f(t), a_2 = f(f(t)), a_3 = f(f(f(t))), \ldots, a_n = f^n(t), \ldots$, where f^n is the composition of f with itself n times. Show that the sequence $\{a_n\}_{n=0}^{\infty}$ is contracting, and hence is convergent.

(b) Let $z = \lim_{n\to\infty} a_n$. Show that $f(z) = z$, that is, z is a fixed point of f. [Hint: Write $f(z) = f\big(\lim_{n\to\infty} a_n\big) = f\big(\lim_{n\to\infty} f^n(t)\big)$. and use the fact that f is continuous.]

(Note: Recall that a **fixed point** of a function f is a point y such that $f(y) = y$. It is clear that a contraction can have at most one fixed point. This problem shows that a contraction always does have a fixed point. Furthermore, if t is any real number, then the sequence $\{f^n(t)\}_{n=0}^{\infty}$ converges to that unique fixed point. This is the **Contraction Mapping Theorem** for $\mathbb{R}$.)

Answer:

(a) Note that $f^{n+1}(t) = f(f^n(t))$ and $f^{n+2}(t) = f(f^{n-1}(t))$. We can calculate

$$\begin{aligned}
|a_{n+2} - a_{n+1}| &= |f^{n+2}(t) - f^{n+1}(t)| \\
&= |f(f^{n+1}(t)) - f(f^n(t))| \\
&\le r|f^{n+1}(t) - f^n(t)| \\
&= r|a_{n+1} - a_n|
\end{aligned}$$

That is, $\{a_n\}_{n=1}^{\infty}$ is contracting with contraction factor r. By the contraction principle, $\{a_n\}_{n=1}^{\infty}$ converges.

(b) Let $z = \lim_{n\to\infty} a_n$. Then

$$\begin{aligned}
f(z) &= f\big(\lim_{n\to\infty} a_n\big) \\
&= \lim_{n\to\infty} f(a_n), \text{ since } f \text{ is continuous} \\
&= \lim_{n\to\infty} f(f^n(t)), \text{ since } a_n = f^n(t) \\
&= \lim_{n\to\infty} f^{n+1}(t) \\
&= \lim_{n\to\infty} a_{n+1} \\
&= \lim_{n\to\infty} a_n \\
&= z
\end{aligned}$$