This is a take-home test. You should not work with anyone on test, and you should not discuss the problems with anyone except your professor. You can consult class notes and the textbook. You should not use other books, the Internet, or any other resources. Be sure to show all your work and explain your reasoning!

1. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 - 6x^2 + 25} = \frac{\pi}{10}$

Hint: $\sqrt{3+4i} = \pm(2+i)$ and $\sqrt{3-4i} = \pm(2-i)$

2. Show that $\displaystyle\int_{-infty}^{\infty} \frac{x\sin(x)}{x^2+4}\,dx = \pi e^{-2}$

3. Show that $\displaystyle\int_{0}^{2\pi} \frac{\sin^2(\theta)}{5+4\cos(\theta)}\,d\theta = \frac{\pi}{4}$

4. Find a linear fractional transformation T such that $T(0) = 1$, $T(1) = 2$ and $T(2) = 0$.

5. Suppose that $g(z)$ is analytic on a neighborhood of a. Show that g has a zero of order n at a if and only if $\frac{1}{g(z)}$ has a pole of order n at a.

6. Suppose that $u(x, y)$ is a harmonic function on some open set. Suppose that v is a harmonic conjugate of u **and** that u is also a harmonic conjugate of v. Show that u and v are constant.

7. **(a)** Suppose D and E are domains and $f\colon D \to E$ is a conformal equivalence. Show that every conformal equivalence from $g\colon D \to E$ is of the form $g = f \circ \lambda$ where $\lambda\colon D \to D$ is an automorphism of D.

(b) Find a conformal equivalence that maps the unit circle, $|z| < 1$, to the circle $|z-2| < 2$. (This is easy!) Then find *all* conformal equivalences from the unit circle to the circle $|z-2| < 2$.

8. Let T be the linear fractional transformation $T(z) = \dfrac{az+b}{cz+d}$, and suppose that $|c| = |d|$. Show that T maps the unit circle to a line. (Hint: We know that T either maps the unit circle to a circle or to a line, and the problem does not ask you to identify the line.)