WEEK 14 LAB

MATH 131 Section 1
April 29, 2014
Due May 2, 2014 at 10:10am
Covering Sections 8.3-8.6

Your Name (Print): ANSWER KEY

Group Member 1: ______________

Group Member 2: ______________

Group Member 3: ______________

Work these problems on separate paper first and use this as the final copy. I will collect one paper from each group. **YOU MUST SHOW ALL WORK TO RECEIVE CREDIT.** Simplify your answers so that you have gathered all like terms, cancelled where possible, and so that there are no negative exponents or fractions within fractions in your final answer. Neatness is a plus!

1. Determine whether the following series are convergent or divergent. If a series is convergent, find the sum (if possible!). If it is divergent, explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=0}^{\infty} e\left(\frac{e}{3}\right)^n$

This is a geometric series with $a = e$ and $r = \frac{e}{3}$. Since $-1 < r < 1$, it is convergent to $\frac{a}{1-r}$.

$$= \frac{e}{1 - \frac{e}{3}}$$

+5

$$= \frac{3e}{3-e}$$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^6}$

$0 < \arctan k < \frac{\pi}{2}$ for all $k \geq 1$ so $0 < \frac{\arctan k}{k^6} < \frac{\pi}{2k^6}$ for all $k \geq 1$

+6

$\displaystyle\sum_{k=1}^{\infty} \frac{\pi}{2k^6} = \frac{\pi}{2}\sum_{k=1}^{\infty} \frac{1}{k^6}$ is a p-series with $p = 6 > 1$ and therefore it is convergent.

Thus by the Direct Comparison Test, $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k^6}$ is convergent.

(c) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$

Let $f(x) = \dfrac{1}{x \ln x \ln(\ln x)}$. Then f is positive and continuous for $x \geq 3$.

$$f'(x) = \frac{d}{dx}\left[x \ln x \ln(\ln x)\right]^{-1} = -\left[x \ln x \ln(\ln x)\right]^{-2} \cdot \left[\cancel{x \ln x} \cdot \frac{1}{\cancel{\ln x}} \cdot \frac{1}{\cancel{x}} + \ln(\ln x)\left[x \cdot \tfrac{1}{x} + \ln x\right]\right]$$

$$= -\left[x \ln x \ln(\ln x)\right]^{-2}\left[1 + \ln(\ln x) + \ln x \ln(\ln x)\right] < 0 \quad \text{for } x \geq 3$$

Thus f is decreasing.

$$\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{x \ln x \ln(\ln x)}\,dx \qquad \begin{aligned} u &= \ln(\ln x) \\ du &= \frac{1}{\ln x} \cdot \frac{1}{x}\,dx \end{aligned} \qquad = \lim_{t\to\infty} \int_{\ln(\ln 3)}^{\ln(\ln t)} \frac{1}{u}\,du$$

+6

$$= \lim_{t\to\infty} \ln|u| \Big]_{\ln(\ln 3)}^{\ln(\ln t)} = \lim_{t\to\infty} \left[\ln|\ln(\ln t)| + \ln|\ln(\ln 3)|\right] = \infty$$

Thus $\displaystyle\int_3^{\infty} \frac{1}{x \ln x \ln(\ln x)}\,dx$ is divergent, so $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ is

divergent by the Integral Test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n \cdot n^2}{n!}$

Note $\dfrac{5^n \cdot n^2}{n!} > 0$ for all $n \geq 1$.

$$\lim_{n\to\infty} \frac{\dfrac{5^{n+1}(n+1)^2}{(n+1)!}}{\dfrac{5^n \cdot n^2}{n!}} = \lim_{n\to\infty} \frac{\cancel{5^n} \cdot 5 \cdot (n+1)^{\cancel{2}}}{\cancel{(n+1)}\,\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{5^n} \cdot n^2} = \lim_{n\to\infty} \frac{5(n+1)}{n^2}$$

+6

$$\underset{\text{power}}{\text{High}} \quad \lim_{n\to\infty} \frac{5n}{n^2} = \lim_{n\to\infty} \frac{5}{n} = 0$$

Since $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{5^n \cdot n^2}{n!}$ is convergent by the

Ratio Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{3^n}$

This is an Alternating series so we must apply the Alternating Series Test.

$a_n = \dfrac{\ln n}{3^n}$ Let $f(x) = \dfrac{\ln x}{3^x}$. Then $f(n) = a_n$ for all integers n.

$$\lim_{n\to\infty} \frac{\ln n}{3^n} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\ln x}{3^x} \stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{3^x \ln 3} = \lim_{x\to\infty} \frac{1}{3^x x \ln 3} = 0 \checkmark$$

$$f'(x) = \frac{\cancel{3^x}\cdot\frac{1}{x} - \ln x\,(\cancel{3^x}\ln 3)}{(3^x)^{\cancel{2}}} = \frac{\frac{1}{x} - \ln x \ln 3}{3^x} = \frac{1 - x\ln x \ln 3}{3^x} < 0$$

+6

when $x\ln x\ln 3 > 1$ which is certainly true for $x \geq 3$. i.e. f is decreasing for $x \geq 3$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{3^n}$ is convergent by the Alternating Series Test.

In the reading for Section 8.6 you learned about **absolute** and **conditional** convergence:

Definition: $\displaystyle\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

$\displaystyle\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, but $\displaystyle\sum_{n=1}^{\infty} |a_n|$ diverges.

You also learned that:

Theorem: If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, so does $\displaystyle\sum_{n=1}^{\infty} a_n$.

This means that we should really first test to see if an alternating series is absolutely convergent, since if it is, then we also know it is convergent! If it is not absolutely convergent, then we can apply the Alternating Series Test to see if it is conditionally convergent.

2. Does $\sum_{n=1}^{\infty} \frac{(-1)^n 4n}{5n^2-7}$ converge conditionally, absolutely, or not at all? Carefully make your argument.

Consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^n 4n}{5n^2-7} \right| = \sum_{n=1}^{\infty} \frac{4n}{5n^2-7}$. compare to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{4n}{5n^2-7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4n}{5n^2-7} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{4n^2}{5n^2-7} \underset{\text{Pow}}{\overset{\text{High}}{=}} \lim_{n\to\infty} \frac{4n^2}{5n^2} = \frac{4}{5} > 0$$

Thus either both series are convergent or both are divergent.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the Harmonic series, which is divergent, by the Limit Comparison Test!

$\sum_{n=1}^{\infty} \frac{4n}{5n^2-7}$ is also divergent. Thus $\sum_{n=1}^{\infty} \frac{(-1)^n 4n}{5n^2-7}$ is <u>NOT</u> absolutely convergent.

+6

$$\lim_{n\to\infty} \frac{4n}{5n^2-7} \underset{\text{Pow}}{\overset{\text{High}}{=}} \lim_{n\to\infty} \frac{4n}{5n^2} = \lim_{n\to\infty} \frac{4}{5n} = 0 \checkmark$$

$$a_{n+1} = \frac{4(n+1)}{5(n+1)^2-7} < \frac{4n}{5n^2-7} = a_n \quad \text{if} \quad (4n+4)(5n^2-7) < 4n[5n^2+10n+5-7]$$

$$\cancel{20n^3} + 20n^2 - \cancel{28n} - 28 < \cancel{20n^3} + 40n^2 + 20n - \cancel{28n}$$

$$20n^2 - 28 < 40n^2 + 20n$$

which is true for all $n \geq 1$.

Thus $\sum_{n=1}^{\infty} \frac{(-1)4n}{5n^2-7}$ is conditionally convergent by the Alternating Series Test.

THE RATIO TEST EXTENSION

It turns out that if we test for absolute convergence using the ratio test, we can tell more than just whether or not the series is absolutely convergent. If the ratio r is actually greater than 1, the series will diverge. We dont even need to check conditional convergence!

The Ratio Test Extension: Assume that $\sum_{n=1}^{\infty} a_n$ is a series with non-zero terms and let $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

1. If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $r > 1$ (including ∞), then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $r = 1$, then the test is inconclusive. The series may converge or diverge.

You will find this extension helpful in the WeBWorK problems due tomorrow and beyond!