LAB Week 8

MATH 130 Section 2
March 14, 2019: Happy Pi Day!
Covering Sections 3.4, 3.5 and 3.7

Your Name (Print): ANSWER KEY

1. Differentiate the following functions and simplify your answers. For (a), see if you can find a trigonometric identity to simplify your result. **Explain every answer through work and/or words.** Be sure to distinguish between the function and its derivative.

(a) $f(x) = \dfrac{\tan x - 1}{\sec x}$

$$f'(x) = \frac{\sec x\,(\sec^2 x - 0) - (\tan x - 1)(\sec x \tan x)}{\sec^2 x} = \frac{\cancel{\sec x}\left[\sec^2 x - \tan^2 x + \tan x\right]}{\sec^{\cancel{2}} x}$$

$$= \frac{\left[\sec^2 x - \tan^2 x\right] + \tan x}{\sec x} = \frac{1 + \tan x}{\sec x}$$

(b) $h(t) = 5e^t(7t^2 - 8t)$

$$h'(t) = 5e^t(14t - 8) + 5e^t\left[7t^2 - 8t\right]$$

$$= 5e^t(14t - 8 + 7t^2 - 8t)$$

$$= 5e^t(7t^2 + 6t - 8)$$

(c) $g(x) = \sqrt{x}\sin x$

$$g'(x) = \sqrt{x}(\cos x) + \frac{1}{2}x^{-1/2}(\sin x)$$

$$= \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}}$$

$$= \frac{2x\cos x + \sin x}{2\sqrt{x}}$$

(d) $y = \dfrac{xe^x}{5x^3+2x-1}$ (Hint: You need more than one rule here!)

$$\frac{dy}{dx} = \frac{(5x^3+2x-1)(xe^x + 1\cdot e^x) - xe^x(15x^2+2)}{(5x^3+2x-1)^2}$$

$$= \frac{e^x(5x^4+2x^2-x+5x^3+\cancel{2x}-1-15x^3+\cancel{-2x})}{(5x^3+2x-1)^2} = \frac{e^x(5x^4-10x^3+2x^2-x-1)}{(5x^3+2x-1)^2}$$

(e) $y = e^x \csc x \cot x$ (Hint: remember the example that we did at the end of group work yesterday!)

$$\frac{dy}{dx} = (e^x\csc x)\left(\frac{d}{dx}\cot x\right) + \left(\frac{d}{dx}e^x\csc x\right)\cot x$$

$$= e^x\csc x(-\csc^2 x) + \left(\frac{d}{dx}e^x\cdot\csc x + e^x\cdot\frac{d}{dx}\csc x\right)\cot x$$

$$= -e^x\csc^3 x + (e^x\csc x + e^x\csc x\cot x)\cot x$$

$$= -e^x\csc^3 x + e^x\csc x\cot x - e^x\csc x\cot^2 x = -e^x\csc x(\csc^2 x - \cot x + \cot^2 x)$$

2. Find the points on the curve $y = \dfrac{2e^x}{x^2+1}$ where the tangent line is horizontal.

$$\frac{dy}{dx} = \frac{(x^2+1)(2e^x) - (2e^x)(2x)}{(x^2+1)^2} = \frac{2e^x(x^2+1-2x)}{(x^2+1)^2}$$

$$= \frac{2e^x(x^2-2x+1)}{(x^2+1)^2}$$

Since the derivative is the slope of the tangent line, and the tangent line is horizontal when its slope is zero, we need to know when

$$\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{2e^x(x^2-2x+1)}{(x^2+1)^2} = 0 \implies 2e^x(x^2-2x+1) = 2e^x(x-1)^2 = 0.$$

Since $e^x > 0$ for all x, $2e^x \neq 0$ for any x, so $\frac{dy}{dx} = 0$ only when $(x-1)^2 = 0$ or when $x = 1$.

$$y(1) = \frac{2e^1}{1^2+1} = \frac{2e}{2} = e$$

Thus the tangent line is horizontal at $(1,e)$.

3. Evaluate $\lim_{x\to -1} \dfrac{\cos(x+1)-1}{x^2-3x-4}$. Be sure to show each step carefully.

(Hint: Use the technique we did in class where we introduced the variable θ.)

$$\lim_{x\to -1} \frac{\cos(x+1)-1}{x^2-3x-4} = \lim_{x\to -1} \frac{\cos(x+1)-1}{(x+1)(x-4)}$$

$$= \lim_{x\to -1} \frac{\cos(x+1)-1}{x+1} \cdot \lim_{x\to -1} \frac{1}{x-4}$$

Let $\theta = x+1$.
Then as $x\to -1$, $\theta \to 0$

$$= \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} \cdot \lim_{x\to -1} \frac{1}{x-4}$$

$$= 0 \cdot \frac{1}{-5}$$

$$= 0$$

4. Evaluate $\lim_{x\to 0} \dfrac{x - \tan 5x}{\tan 6x}$. Be sure to show each step carefully.

$$\lim_{x\to 0} \frac{x - \tan 5x}{\tan 6x} = \lim_{x\to 0} \frac{\left(x - \frac{\sin 5x}{\cos 5x}\right) \cdot \frac{1}{x}}{\left(\frac{\sin 6x}{\cos 6x}\right) \cdot \frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{x}{x} - \frac{\sin 5x}{x} \cdot \frac{1}{\cos 5x}}{\frac{\sin 6x}{x} \cdot \frac{1}{\cos 6x}}$$

$$= \frac{\lim_{x\to 0} 1 \to 1 \; - \; \lim_{x\to 0} \frac{\sin 5x}{x} \cdot \frac{5}{5} \cdot \lim_{x\to 0} \frac{1}{\cos 5x} \to 1}{\lim_{x\to 0} \frac{\sin 6x}{x} \cdot \frac{6}{6} \cdot \lim_{x\to 0} \frac{1}{\cos 6x} \to 1}$$

Let $\theta_1 = 5x$ & $\theta_2 = 6x$. Then as $x \to 0$, $\theta_1, \theta_2 \to 0$

$$= \frac{1 - \lim_{\theta_1\to 0} \frac{\sin\theta_1}{\theta_1} \to 1 \; \cdot 5 \cdot 1}{\lim_{\theta_2\to 0} \frac{\sin\theta_2}{\theta_2} \to 1 \; \cdot 6 \cdot 1}$$

$$= \frac{1-5}{6} = \frac{-4}{6} = -\frac{2}{3}$$

5. Suppose $f'(2) = 3$, $g'(2) = -4$, $f(2) = -1$, $g(2) = 2$, $f'(1) = 4$ and $g'(-1) = -2$.

(a) If $h(x) = f(g(x))$, find $h'(2)$. Be sure to show each step carefully.

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$h'(2) = f'(g(2)) \cdot g'(2)$$
$$= f'(2) \cdot (-4)$$
$$= 3 \cdot (-4)$$
$$= -12$$

(b) If $k(x) = g(f(x))$, find $k'(2)$. Be sure to show each step carefully.

$$k'(x) = g'(f(x)) \cdot f'(x)$$
$$k'(2) = g'(f(2)) \cdot f'(2)$$
$$= g'(-1) \cdot (3)$$
$$= (-2)(3)$$
$$= -6$$

6. Given $y = \cos 6x$, find $\frac{d^{67}y}{dx^{67}}$. Explain your work carefully.

$$\frac{dy}{dx} = -\sin 6x \cdot 6 = -6\sin 6x$$
$$\frac{d^2y}{dx^2} = -\cos 6x \cdot 6 \cdot 6 = -6^2 \cos 6x$$
$$\frac{d^3y}{dx^3} = \sin 6x \cdot 6 \cdot 6 \cdot 6 = 6^3 \sin 6x$$
$$\frac{d^4y}{dx^4} = \cos 6x \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^4 \cos 6x$$

* Note this is a constant times the original function.

The constant will always be 6^n where n is the derivative we are taking. The function rotates on a cycle of four: $\cos 6x$, $-\sin 6x$, $-\cos 6x$, $\sin 6x$, with the first derivate starting with $-\sin 6x$. We must find the remainder when 67 is divided by four:

$67 \div 4 = 16$ R 3. Thus $\frac{d^{67}y}{dx^{67}} = 6^{67} \cdot \sin 6x = 6^{67} \sin 6x$.

7. Fill in the following table to determine the derivatives of the given composite functions:

Composite: $y = f(g(x))$	Outside: $y = f(u)$	Outside Deriv: $\frac{dy}{du} = f'(u)$	Inside: $u = g(x)$	Inside Deriv: $\frac{du}{dx} = g'(x)$	Chain Rule: $\frac{dy}{du} \cdot \frac{du}{dx}$ $f'(u)\frac{du}{dx} = f'(g(x))g'(x)$
$\sin(3x^2+1)$	$\sin u$	$\cos u$	$3x^2+1$	$6x$	$\cos u \cdot 6x = 6x\cos(3x^2+1)$
$\cos(e^{2x})$	$\cos w$	$-\sin w$	3 functions! e^u $2x$	$\frac{dw}{du} = e^u$ $\frac{du}{dx} = 2$	$-\sin w \cdot e^u \cdot 2$ $= -2e^{2x}\sin(e^{2x})$
$\tan(\sin x)$	$\tan u$	$\sec^2 u$	$\sin x$	$\cos x$	$\sec^2 u \cdot \cos x$ $= \sec^2(\sin x)\cos x$
$(3x^2+1)^5$	u^5	$5u^4$	$3x^2+1$	$6x$	$5u^4 \cdot 6x$ $= 5(3x^2+1)^4 \cdot 6x$
$\sin^5 x =$ $(\sin x)^5$	u^5	$5u^4$	$\sin x$	$\cos x$	$5u^4 \cdot \cos x$ $= 5\sin^4 x \cos x$
$\sqrt{3x^2+1}$ $= (3x^2+1)^{1/2}$	$u^{1/2}$	$\frac{1}{2}u^{-1/2}$	$3x^2+1$	$6x$	$\frac{1}{2}u^{-1/2} \cdot 6x$ $= \frac{6x}{2\sqrt{3x^2+1}} = \frac{3x}{\sqrt{3x^2+1}}$
$\frac{1}{3x^2+1} =$ $(3x^2+1)^{-1}$	u^{-1}	$-u^{-2}$	$3x^2+1$	$6x$	$-u^{-2} \cdot 6x$ $= \frac{-6x}{(3x^2+1)^2}$
$\left(\frac{x+5}{3x+1}\right)^4$	u^4	$4u^3$	$\frac{x+5}{3x+1}$	$\frac{(3x+1)(1)-(x+5)(3)}{(3x+1)^2}$ $= \frac{3x+1-3x-15}{(3x+1)^2}$ $= \frac{-14}{(3x+1)^2}$	$4u^3 \cdot \frac{-14}{(3x+1)^2}$ $= \frac{(x+5)^3}{(3x+1)^3} \cdot \frac{-56}{(3x+1)^2}$ $= \frac{-56(x+5)^3}{(3x+1)^5}$

8. Now try differentiating the following functions and simplify your answers. **Explain every answer through work and/or words.** Be sure to distinguish between the function and its derivative.

(a) $g(x) = e^{\sin x}\sqrt{x}$

$$g'(x) = e^{\sin x} \cdot \frac{1}{2}x^{-1/2} + \sqrt{x} \cdot e^{\sin x} \cdot \cos x$$

$$= \frac{e^{\sin x}}{2\sqrt{x}} + e^{\sin x}(\cos x)\sqrt{x}$$

(b) $y = \csc(10x^5 + 17)$

$$\frac{dy}{dx} = -\csc(10x^5+17)\cot(10x^5+17) \cdot 50x^4$$

$$= -50x^4 \csc(10x^5+17)\cot(10x^5+17)$$

(c) $y = \sqrt{x^3 - \cos^9(4x)}$

$$\frac{dy}{dx} = \frac{1}{2}(x^3 - [\cos(4x)]^9)^{-1/2} \cdot (3x^2 - 9[\cos(4x)]^8 \cdot (-\sin(4x)) \cdot 4)$$

$$= \frac{3x^2 + 36\cos^8(4x)\sin(4x)}{2\sqrt{x^3 - \cos^9(4x)}}$$