WEEK 6 LAB

MATH 131: Calculus II
February 27, 2020
Covering Sections 5.5, 6.1-6.4

Your Name (Print): ANSWER KEY

1. Suppose that the velocity of an object measured in meters/sec is $v(t) = t^2 - 6t + 8$ for $0 \leq t \leq 5$.

(a) Find the displacement over the given interval.

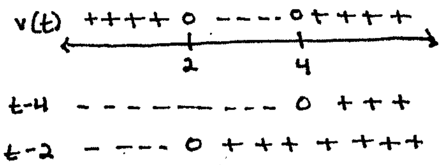
$$\text{Displacement} = \int_0^5 (t^2 - 6t + 8)\,dt = \frac{t^3}{3} - 3t^2 + 8t\Big]_0^5 = \left(\frac{125}{3} - 75 + 40\right) - 0$$

$$= \frac{125}{3} - 35 = \frac{125 - 105}{3}$$

$$= \frac{20}{3} \text{ meters}$$

(b) Find the distance traveled over the same interval.

$$t^2 - 6t + 8 = (t-4)(t-2)$$



$$\text{Thus } |v(t)| = \begin{cases} t^2 - 6t + 8 & \text{if } t \leq 2 \\ -(t^2 - 6t + 8) & \text{if } 2 < t \leq 4 \\ t^2 - 6t + 8 & \text{if } t > 4 \end{cases}$$

$$\text{Distance} = \int_0^5 |v(t)|\,dt = \int_0^2 (t^2 - 6t + 8)\,dt + \int_2^4 -(t^2 - 6t + 8)\,dt + \int_4^5 (t^2 - 6t + 8)\,dt$$

$$= \cdots = \frac{28}{3} \text{ meters}$$

2. When records were first kept, the population of a rural town was 250 people ($t = 0$). Since then the population has grown at a rate $P'(t) = 30(1 + \sqrt{t})$, where t is measured in years.

(a) What was the population at time $t = 20$ years?

$$P(20) = P(0) + \int_0^{20} 30(1 + \sqrt{t})\,dt = 250 + \left[30\left(t + \frac{2}{3}t^{3/2}\right)\right]_0^{20}$$

$$= 250 + 30\left[\left(20 + \frac{2}{3}(20)^{3/2}\right) - 0\right] = 250 + 30\left(20 + \frac{2}{3}(2\sqrt{5})^3\right)$$

$$= 250 + 600 + \overset{10}{\cancel{30}} \cdot \frac{2}{\cancel{3}} \cdot 40\sqrt{5} = 850 + 800\sqrt{5} \approx 850 + 1788.9$$

so ≈ 2639 people (we must have whole people :))

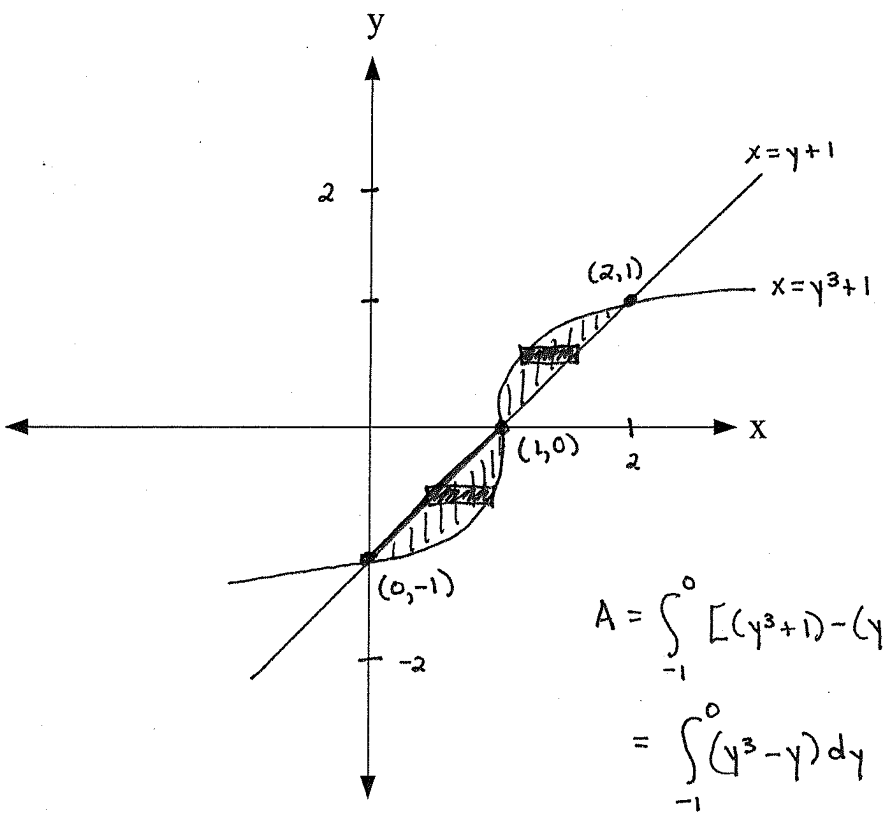
(b) Find the population $P(t)$ for an arbitrary time $t \geq 0$.

$$P(t) = P(0) + \int_0^t 30(1 + \sqrt{x})\,dx = 250 + \left[30\left(x + \frac{2}{3}x^{3/2}\right)\right]_0^t$$

$$= 250 + 30\left(t + \frac{2}{3}t^{3/2}\right) \text{ people}$$

* There is more than one way to do this question! :)

3. Find the area of the region bounded by $x = y^3 + 1$ and $x = y + 1$.



$y^3 + 1 = y + 1$

$y^3 - y = 0$

$y(y^2 - 1) = 0$

$y(y+1)(y-1) = 0$

$y = 0, y = -1, y = 1$

$(1,0) ; (0,-1) ; (2,1)$

$$A = \int_{-1}^{0} [(y^3+1) - (y+1)]\,dy + \int_{0}^{1} [(y+1) - (y^3+1)]\,dy$$

$$= \int_{-1}^{0} (y^3 - y)\,dy + \int_{0}^{1} (y - y^3)\,dy$$

$$= \left.\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{0} + \left.\frac{y^2}{2} - \frac{y^4}{4}\right]_{0}^{1}$$

$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{1}{2} - \frac{1}{4}\right) - 0\right]$$

$$= 1 - \frac{1}{2}$$

$$= \boxed{\frac{1}{2}}$$

4. Find the area of the region bounded by $y = e^x$, $x + 3y = 3$, and $x = 3$. (Leave e's in your answer.)

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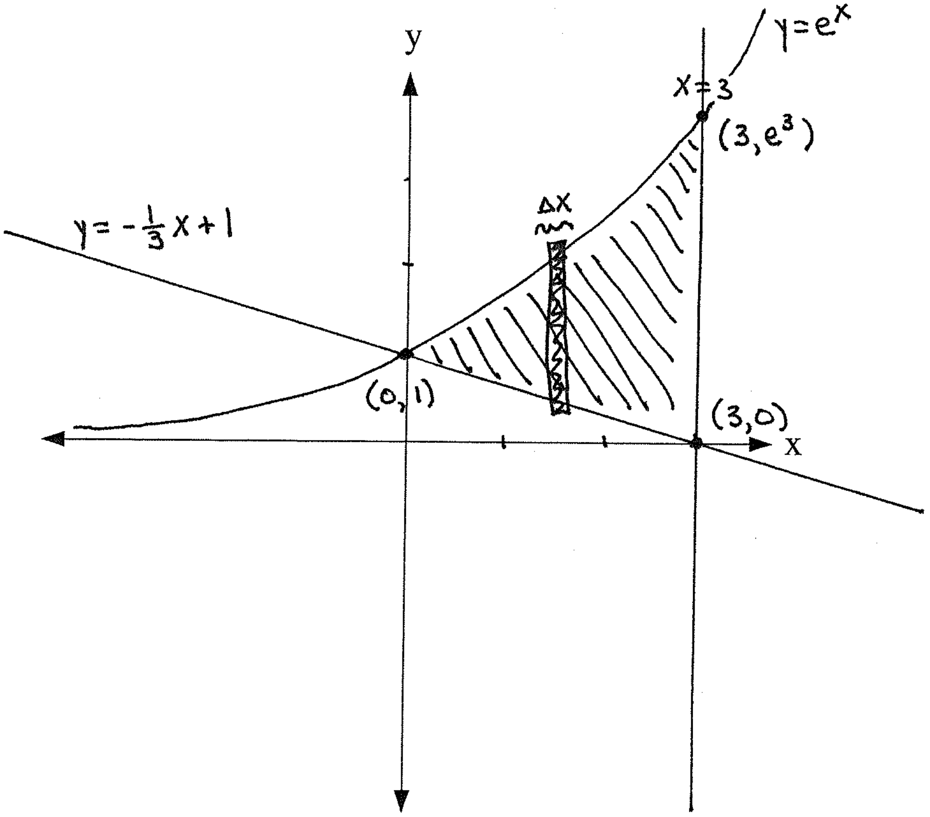


$$A = \int_0^3 \left(e^x - \left(-\tfrac{1}{3}x + 1\right)\right) dx$$

$$= \int_0^3 \left(e^x + \tfrac{1}{3}x - 1\right) dx$$

$$= e^x + \tfrac{1}{3}\cdot\tfrac{1}{2}x^2 - x\Big]_0^3 \quad = e^x + \tfrac{1}{6}x^2 - x\Big]_0^3$$

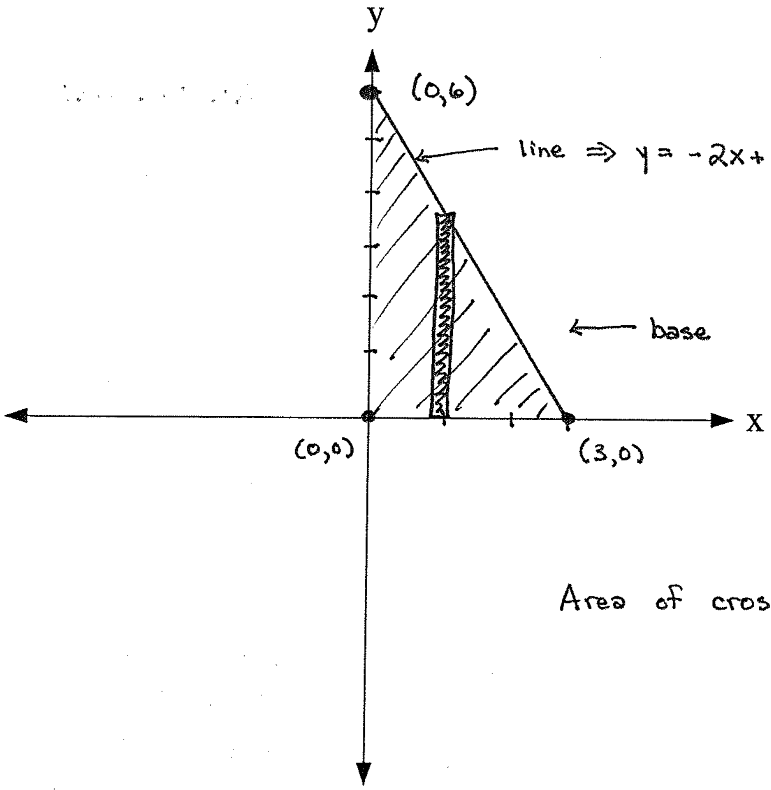
$$= \left(e^3 + \tfrac{9}{6} - 3\right) - \left(e^0 + \tfrac{0}{6} - 0\right)$$

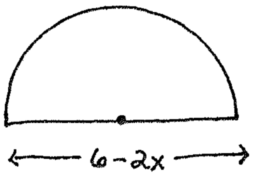
$$= e^3 + \tfrac{3}{2} - 3 - 1$$

$$= e^3 + \tfrac{3}{2} - 4 \quad = e^3 + \tfrac{3}{2} - \tfrac{8}{2}$$

$$= \boxed{e^3 - \tfrac{5}{2}} \qquad \approx 17.5855$$

5. Find the volume of the solid whose base is the triangle with vertices $(0,0)$, $(3,0)$ and $(0,6)$, and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles.





$$\text{Area of cross section} = \frac{1}{2}\pi r^2$$

$$= \frac{1}{2}\pi\left[\frac{6-2x}{2}\right]^2$$

$$= \frac{\pi}{2}\left[9-6x+x^2\right]$$

$$V = \int_0^3 A(x)\,dx$$

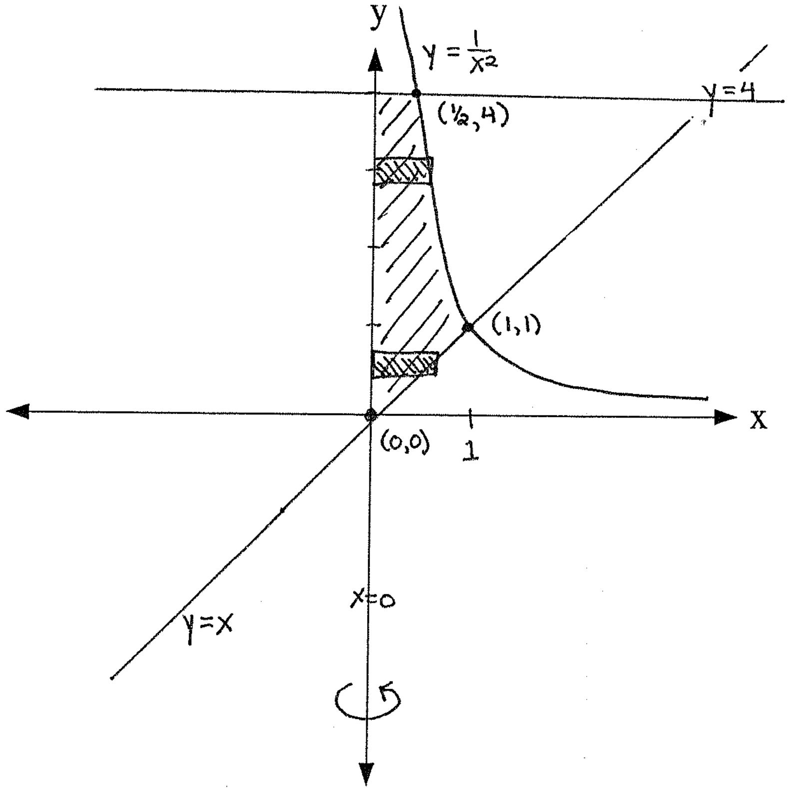
$$= \int_0^3 \frac{\pi}{2}\left[9-6x+x^2\right]dx$$

$$= \frac{\pi}{2}\left[9x-3x^2+\frac{x^3}{3}\right]_0^3$$

$$= \frac{\pi}{2}\left[\left(\cancel{27}-\cancel{27}+\frac{27}{3}\right)-0\right]$$

$$= \boxed{\frac{9\pi}{2}}$$

6 Find the volume of the solid obtained by revolving the region bounded by the curves $y = \frac{1}{x^2}$, $y = x$, $x = 0$ and $y = 4$ about the y-axis.



$$y = \frac{1}{x^2} \Rightarrow x = \pm\frac{1}{\sqrt{y}}$$

+6

$$V = \int_0^1 \pi y^2 \, dy + \int_1^4 \pi \left(\frac{1}{\sqrt{y}}\right)^2 dy$$

$$= \pi \int_0^1 y^2 \, dy + \pi \int_1^4 \frac{1}{y} \, dy$$

$$= \pi \left[\frac{y^3}{3}\right]_0^1 + \pi \ln|y| \Big]_1^4$$

$$= \pi \left[\frac{1}{3} - 0\right] + \pi \left[\ln 4 - \overset{0}{\cancel{\ln 1}}\right]$$

$$= \pi \left[\frac{1}{3} + \ln 4\right]$$

Shell :

$$V = \int_0^{1/2} 2\pi x (4 - x) \, dx + \int_{1/2}^1 2\pi x \left(\frac{1}{x^2} - x\right) dx$$

$$= 2\pi \int_0^{1/2} (4x - x^2) \, dx + 2\pi \int_{1/2}^1 \left(\frac{1}{x} - x^2\right) dx$$

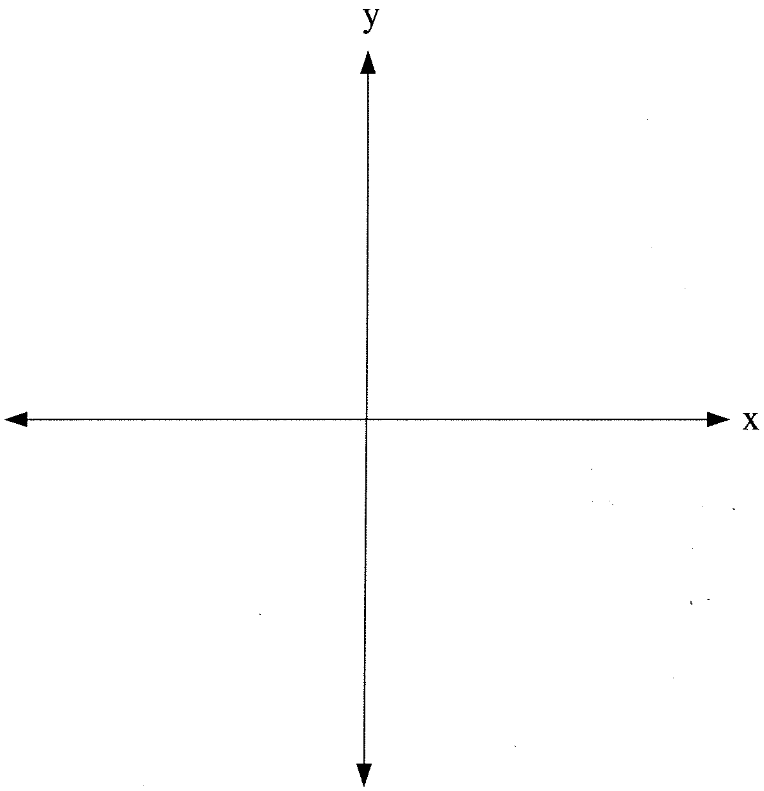
$$= 2\pi \left[2x^2 - \frac{x^3}{3}\right]_0^{1/2} + 2\pi \left[\ln|x| - \frac{x^3}{3}\right]_{1/2}^1$$

$$= 2\pi \left[2\left(\frac{1}{4}\right) - \cancel{\frac{1}{3}\left(\frac{1}{8}\right)} - 0\right] + 2\pi \left[\ln 1 - \frac{1}{3} - \ln\left|\frac{1}{2}\right| + \cancel{\frac{1}{3}\left(\frac{1}{8}\right)}\right]$$

$$= 2\pi \left[\frac{1}{2} - \frac{1}{3}\right] + 2\pi \ln 2$$

$$= \pi \left[\frac{1}{3} + 2\ln 2\right]$$

7. Find the volume of the solid obtained by revolving the region bounded by the curves $y = \frac{1}{x^2}$, $y = x$, $y = 0$ and $x = 3$ about the x-axis.



ANS: $\frac{53}{81}\pi$

8. Evaluate the following integrals. (Hint: Complete the square!)

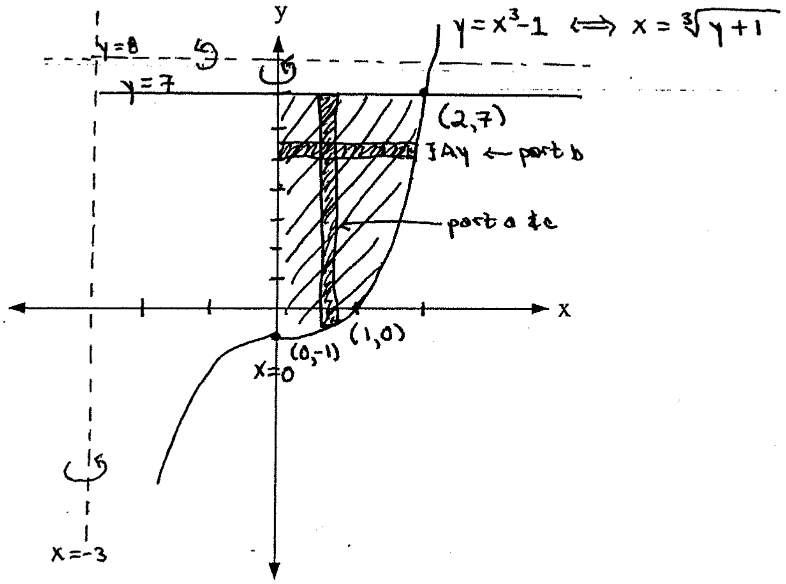
(a) $\displaystyle\int \frac{1}{(x+7)\sqrt{4x^2+56x+180}}dx$

ANS: $\frac{1}{4}\,\text{arcsec}\left(\frac{x+7}{2}\right) + C$

(b) $\displaystyle\int \frac{1}{x^2+6x+16}dx$

ANS: $\frac{1}{\sqrt{7}}\arctan\left(\frac{x+3}{\sqrt{7}}\right) + C$

9. Sketch the region bounded by the curves $y = x^3 - 1$, $y = 7$ and $x = 0$.



(a) Set up an integral for finding the volume of the solid obtained by revolving the region about the $y-axis$, but do not evaluate it. Include an estimating rectangle in your diagram above, and state which method you chose to use.

$$V = \int_0^2 2\pi x \left(7 - (x^3 - 1)\right) dx \quad ; \quad \text{shell method}$$

(b) Set up an integral for finding the volume of the solid obtained by revolving the region about $y = 8$, but do not evaluate it. Include an estimating rectangle in your diagram above, and state which method you chose to use.

$$V = \int_{-1}^{7} 2\pi (8-y)\left(\sqrt[3]{y+1}\right) dy \quad ; \quad \text{shell method}$$

(c) Find the volume of the solid obtained by revolving the region about the line $x = -3$. That is, set up AND evaluate it! Include an estimating rectangle in your diagram above, and state which method you chose to use.

$$V = \int_0^2 2\pi (3+x)\left(7 - (x^3 - 1)\right) dx = \int_0^2 2\pi (3+x)(8 - x^3)\, dx$$

$$= 2\pi \int_0^2 (24 - 3x^3 + 8x - x^4)\, dx = 2\pi \left[24x - \tfrac{3}{4}x^4 + 4x^2 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi \left[\left(48 - 12 + 16 - \tfrac{32}{5}\right) - 0\right] = 2\pi \left[52 - \tfrac{32}{5}\right]$$

$$= 2\pi \left[\frac{260 - 32}{5}\right] = \frac{456\pi}{5}$$