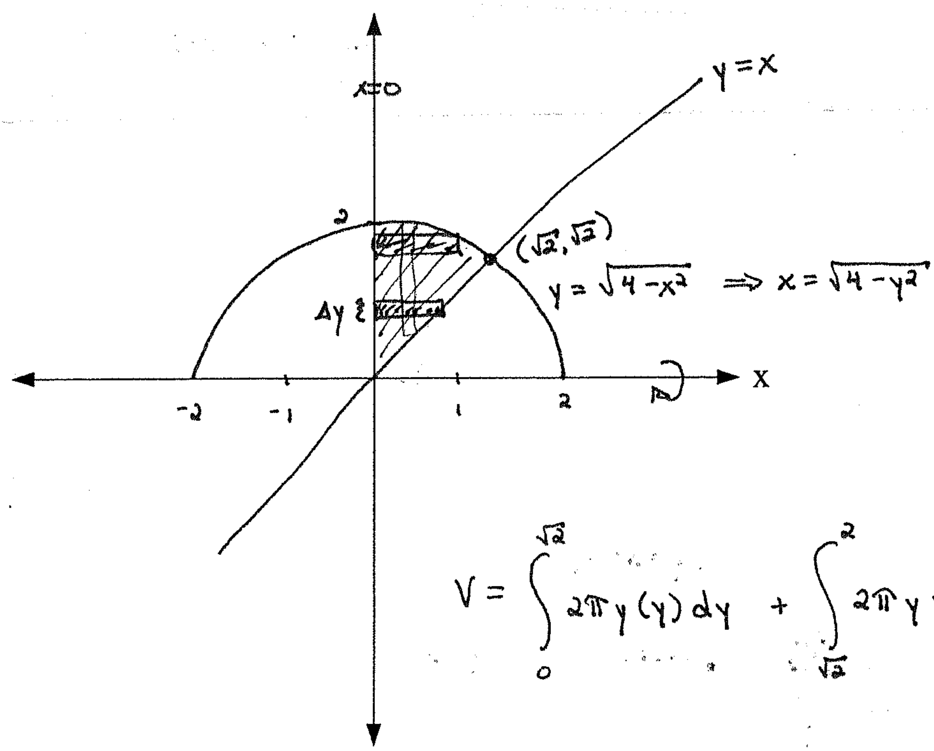
WEEK 7 LAB

MATH 131: Calculus II
March 5, 2020
Covering Sections 6.3-6.~~5~~6

Your Name (Print): ANSWER KEY

1. (a) **Using the cylindrical shell method**, find the volume of the solid obtained by revolving the region bounded by the curves $y = x$, $y = \sqrt{4 - x^2}$ and $x = 0$ about the x-axis.



$$V = \int_0^{\sqrt{2}} 2\pi y(y)\,dy + \int_{\sqrt{2}}^{2} 2\pi y\sqrt{4-y^2}\,dy$$

$$= 2\pi\int_0^{\sqrt{2}} y^2\,dy + 2\pi\int_{\sqrt{2}}^{2} y\sqrt{4-y^2}\,dy \qquad \begin{aligned} u &= 4-y^2 \\ du &= -2y\,dy \\ -\tfrac{1}{2}du &= y\,dy \end{aligned}$$

$$= 2\pi\cdot\frac{y^3}{3}\Big]_0^{\sqrt{2}} + -\pi\int_2^0 u^{1/2}\,du = \frac{2\pi}{3}\left[(\sqrt{2})^3 - 0\right] + \pi\frac{2}{3}u^{3/2}\Big]_0^2$$

$$= \frac{2\pi}{3}\left[(\sqrt{2})^3\right] + \frac{2\pi}{3}\left[(\sqrt{2})^3 - 0\right] = \frac{4\pi}{3}\left[\sqrt{2}\right]^3$$

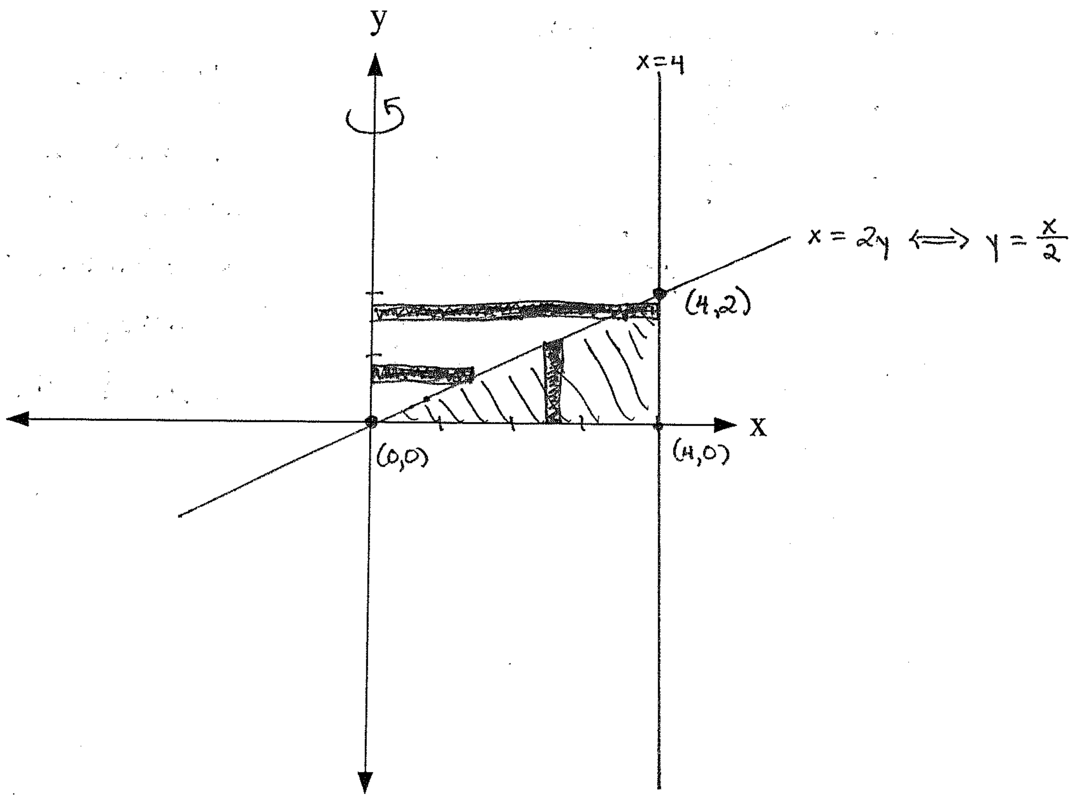
$$= \frac{8\pi\sqrt{2}}{3} \text{ or } \frac{8\sqrt{2}\pi}{3}$$

(b) Is this the preferable method? Why or why not?

No. With disk method we would need only one integral.

2. Give a geometric argument that explains why the following integrals have equal values. Use both a sketch and **full sentences** for a complete argument. Think about what question might have been asked for which these integrals could be part of the solution.

$$\pi \int_0^2 [16 - (2y)^2]dy = 2\pi \int_0^4 x\left(\frac{x}{2}\right)dx$$



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Consider the region bounded by $x = 2y$, $x = 4$ and $y = 0$. Rotate this region about the y-axis. To find the volume of the resulting solid, we would consider a disk of radius four minus a disk of radius $2y$ as y varies from zero to two, giving us:

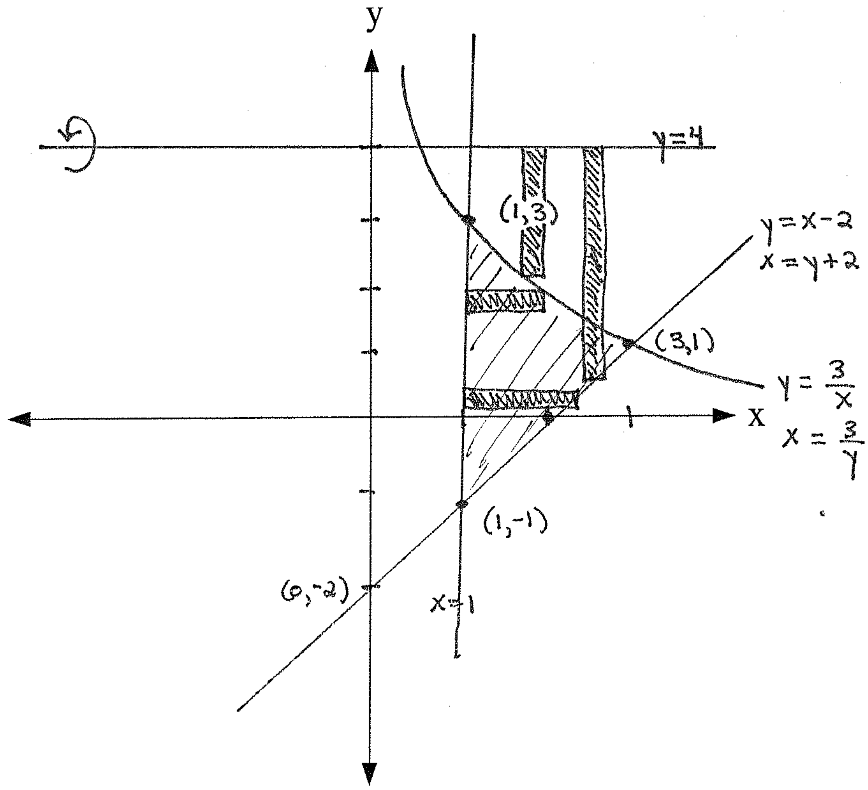
$$V = \int_0^2 \left(\pi 4^2 - \pi (2y)^2\right) dy$$

To find the volume of the resulting solid using the shell method, we would consider shells with average radius x and height $\frac{x}{2}$ as x varies from zero to four, which gives us:

$$V = \int_0^4 2\pi x \left(\frac{x}{2}\right) dx.$$

Hence these two integrals are equal.

3. Sketch the region bounded by $y = \frac{3}{x}$, $y = x - 2$ and $x = 1$. Set up the integrals used to find the volume obtained by revolving the region about $y = 4$ using (a) the disk method AND (b) the shell method. **DO NOT EVALUATE THE INTEGRALS!.**



$$\frac{3}{x} = x-2$$

$$3 = x^2 - 2x$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$

$$(3,1) \ \& \ (-1,-3)$$

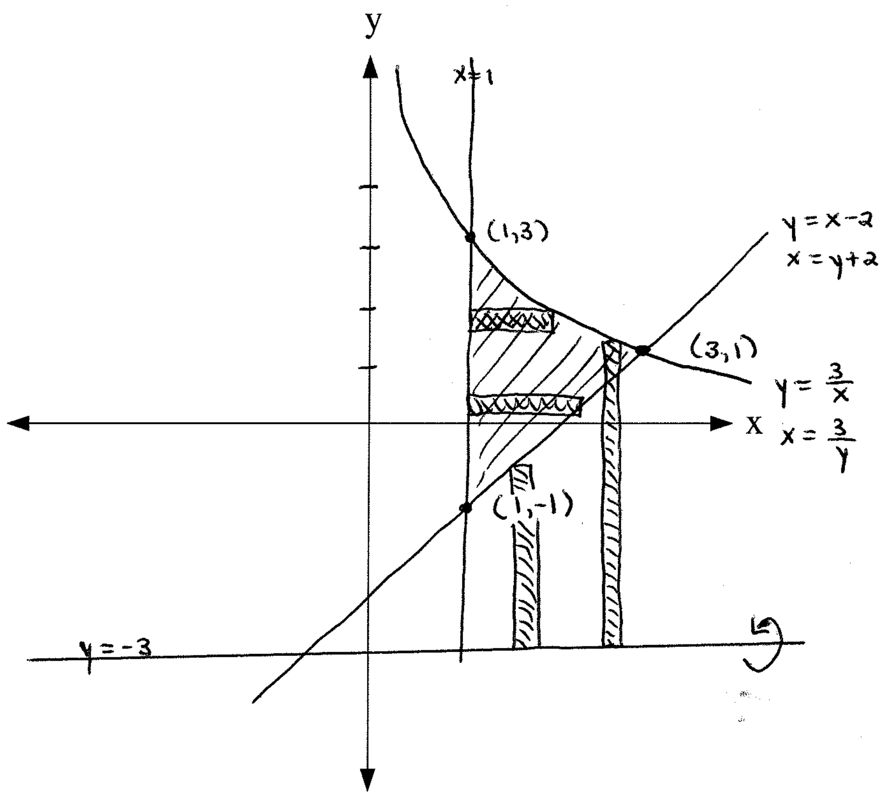
a) Disk Method

$$V = \int_1^3 \pi \left[\left(4-(x-2)\right)^2 - \left(4 - \frac{3}{x}\right)^2 \right] dx$$

b) Shell Method

$$V = \int_{-1}^{1} 2\pi (4-y)\left((y+2)-1\right) dy + \int_1^3 2\pi (4-y)\left(\frac{3}{y} - 1\right) dy$$

4. Repeat question three, except rotate the region about $y = -3$.



a) Disk Method

$$V = \int_1^3 \pi \left[\left(\frac{3}{x} - (-3)\right)^2 - \left((x-2) - (-3)\right)^2\right] dx$$

b) Shell Method

$$V = \int_{-1}^{1} 2\pi (y+3)\left((y+2) - 1\right) dy + \int_1^3 2\pi (y+3)\left(\frac{3}{y} - 1\right) dy$$

5. Find the length of the curve $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on the interval $[1, 3]$.

$$y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$$

$$y' = x^3 - \frac{1}{4}x^{-3} = x^3 - \frac{1}{4x^3}$$

$$L = \int_1^3 \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{1 + x^6 - \frac{1}{2} + \frac{1}{16x^6}}\, dx$$

$$= \int_1^3 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16x^6}}\, dx$$

$$= \int_1^3 \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx$$

$$= \int_1^3 \left(x^3 + \frac{1}{4}x^{-3}\right) dx$$

$$= \frac{1}{4}x^4 + \frac{1}{4}\cdot\frac{1}{-2}x^{-2}\Big]_1^3$$

$$= \left(\frac{81}{4} - \frac{1}{72}\right) - \left(\frac{1}{4} - \frac{1}{8}\right)$$

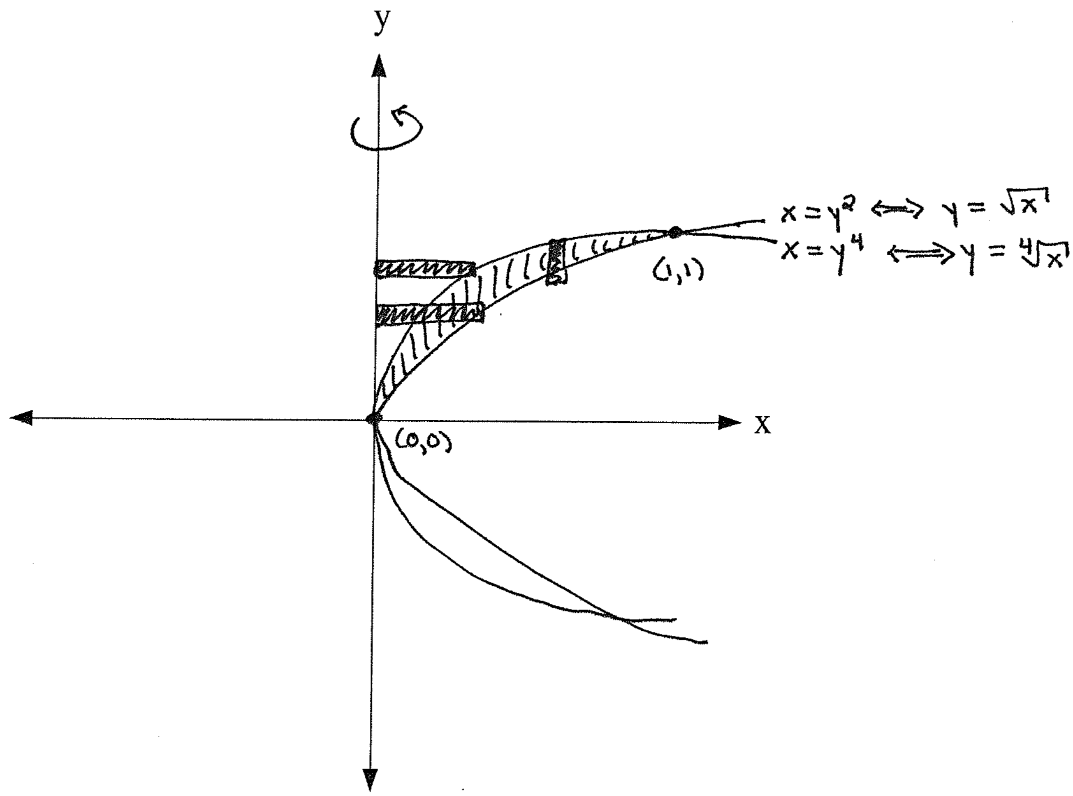
$$= \frac{80}{4} + \frac{8}{72}$$

$$= 20 + \frac{1}{9}$$

$$= \frac{181}{9}$$

6. The integral $\pi \int_0^1 (y^4 - y^8)dy$ represents the volume of a solid found using the disk method.

(a) Describe the solid. Use both a sketch and full sentences for a complete argument.



The solid is formed by rotating the region bound by $y = \sqrt{x}$ and $y = \sqrt[4]{x}$ about the y-axis.

$$V = \pi \int_0^1 [(y^2)^2 - (y^4)^2]\,dy$$

(b) Set up the integral to find the volume of the same solid using the shell method. **Do not evaluate the integrals!**

$$V = \int_0^1 2\pi x \left(\sqrt[4]{x} - \sqrt{x}\right)dx$$

TWO QUESTIONS RELATED TO u-SUBSTITUTION!

7. Evaluate $\int_0^2 x\sqrt{16-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area. Be sure to show your work and explain how you are interpreting in terms of an area.

Let $u = x^2$ When $x=0$, $u=0$

$du = 2x\,dx$ $x=2$, $u=4$

$\frac{1}{2}du = x\,dx$

Thus we have

$$\frac{1}{2}\underbrace{\int_0^4 \sqrt{16-u^2}\,du}_{\frac{1}{4}\text{ circle centered at }(0,0)\text{ with radius }4} = \frac{1}{2}\left(\frac{1}{4}\pi(4)^2\right) = 2\pi$$

8. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^1 2xf(4x^2)\,dx$.

Let $u = 4x^2$

$du = 8x\,dx$

$\frac{1}{4}du = 2x\,dx$

When $x=0$, $u=0$

$x=1$, $u=4$

$$\Rightarrow \int_0^1 2xf(4x^2)dx = \frac{1}{4}\int_0^4 f(u)\,du = \frac{1}{4}(10) = \frac{5}{2}$$

9. Find the volume of the solid obtained by revolving the region bounded by the curve $y=\sqrt{x}+2$, the line tangent to this curve at $x=1$ and $x=0$ about the line $y=1$.

Answer: $\frac{\pi}{4}$

Hint: Start by finding the slope of the tangent line at $x=1$, then set up the equation of the tangent line.

10. Find the area of the surface of revolution obtained by rotating the curve $x = 1 + 2y^2$ for $1 \le y \le 2$ about the x-axis.

$$x = 1 + 2y^2 \quad \Rightarrow \quad \frac{dx}{dy} = 4y$$

$$SA = \int_1^2 2\pi y \sqrt{1 + (4y)^2}\, dy$$

$$= 2\pi \int_1^2 y \sqrt{1 + 16y^2}\, dy$$

Let $u = 1 + 16y^2$
$du = 32y\,dy \Rightarrow \frac{1}{16} du = 2y\,dy$
If $y = 1$, then $u = 1 + 16 = 17$
$y = 2$, then $u = 1 + 16(4) = 65$

$$= \frac{\pi}{16} \int_{17}^{65} u^{1/2}\, du$$

$$= \frac{\pi}{16} \cdot \frac{2}{3} u^{3/2} \Big]_{17}^{65}$$

$$= \frac{\pi}{24}\left[65^{3/2} - 17^{3/2}\right] \quad \text{or} \quad \frac{\pi}{24}\left[65\sqrt{65} - 17\sqrt{17}\right]$$

11. Find the length of the curve $y = \frac{3}{2}x^{\frac{2}{3}} + 4$ on the interval [1,27].

$$\frac{dy}{dx} = \frac{3}{2} \cdot \frac{2}{3} x^{-1/3} = x^{-1/3} \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{x^{2/3}} = \frac{x^{2/3} + 1}{x^{2/3}}$$

$$L = \int_1^{27} \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}}\, dx = \int_1^{27} \frac{\sqrt{x^{2/3} + 1}}{\sqrt{x^{2/3}}}\, dx$$

$$= \int_1^{27} \frac{\sqrt{x^{2/3} + 1}}{x^{1/3}}\, dx$$

Let $u = x^{2/3} + 1$
$du = \frac{2}{3} x^{-1/3} dx$
$\frac{3}{2} du = x^{-1/3} dx$

When $x = 1$, $u = 2$
$x = 27$, $u = 27^{2/3} + 1 = 10$

$$= \frac{3}{2} \int_2^{10} u^{1/2}\, du$$

$$= \frac{3}{2} \cdot \frac{2}{3} u^{3/2} \Big]_2^{10}$$

$$= 10^{3/2} - 2^{3/2} \quad \text{or} \quad 10\sqrt{10} - 2\sqrt{2}$$