Reviewing Calculus I: ANSWER KEY

MATH 131: Calculus II January 22, 2020

This worksheet covers **some** of the material from first semester calculus that you will need to have at your fingertips for this semester. If you have significant difficulties completing any part of it, please see me immediately. We will be working on these problems for part of class on Monday and Wednesday. It is recommended that you complete the worksheet on your own if you are not able to finish it in class. Answers will be posted at the end of the week.

1. Evaluate $\lim_{x\to 1} \frac{4x^2+3}{6x^3-8}$. Why is this easy? What kind of function are you taking the limit of?

$\lim_{x\to 1} \frac{4x^2+3}{6x^3-8} = \frac{7}{-2}$; This is the limit of a rational function, and rational functions are continuous on their domains. Since 1 is in its domain, the function is continuous at $x = 1$ and we can just plug in.

2. (a) Evaluate $\lim_{t\to 4} \frac{t-4}{t^2-16}$. Show each step. (There are at least two ways to do this! Can you find two ways?)

One way: $\lim_{t\to 4} \frac{t-4}{t^2-16} = \lim_{t\to 4} \frac{t-4}{(t-4)(t+4)} = \lim_{t\to 4} \frac{1}{t+4} = \frac{1}{8}$; Another way: $\lim_{t\to 4} \frac{t-4}{t^2-16} = \lim_{t\to 4} \frac{1}{2t} = \frac{1}{8}$

(b) Compare the functions $f(t) = \frac{t-4}{t^2-16}$ and $g(t) = \frac{1}{t+4}$. Carefully determine whether or not they are the same function. Explain your answer.

The functions $f(t)$ and $g(t)$ are NOT the same function since 4 is in the domain of g, but not in the domain of f. They have different domains and if two functions are equal they must have the same domains.

3. In Calculus II, most of the limits we will use will be limits at infinity. The following theorem makes it easier if we are dealing with rational functions.
Theorem: Suppose k,n,a_0,...,a_k,b_0,...,b_n are constants such that $a_k \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x\to\infty} \frac{a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0} = \begin{cases} \frac{a_k}{b_n} & \text{if } k = n \\ 0 & \text{if } n > k \\ \pm\infty & \text{if } k > n \end{cases}$$

Note that the sign of case three (when $k > n$) depends on the sign of $\frac{a_k}{b_n}$.

Evaluate the following using the theorem and **explain your answers.**

(a) $\lim_{x\to\infty} \frac{-3x^8+4x^2-8}{-12x^5-7x^3+6x}$

$\lim_{x\to\infty} \frac{-3x^8+4x^2-8}{-12x^5-7x^3+6x} = \infty$ since $k = 8$ is greater than $n = 5$ and $\frac{a_k}{b_n} = \frac{-3}{-12}$ is positive

(b) $\lim_{x\to\infty} \frac{13x^6-7x^4+5x^3}{-2x^8+5x^2-156}$

$$\lim_{x\to\infty} \frac{13x^6-7x^4+5x^3}{-2x^8+5x^2-156} = 0 \text{ since } n = 8 \text{ is greater than } k = 6$$

4. Continuity is a property we are very much interested in our functions having. What is it?

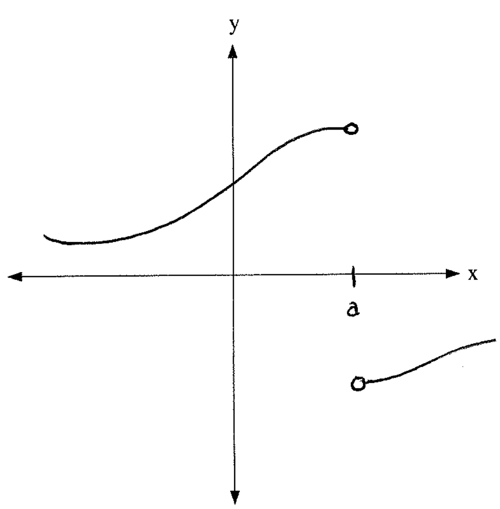
(a) Finish the statement: We say a function $f(x)$ is **continuous** at a point "a" if all of the following hold:

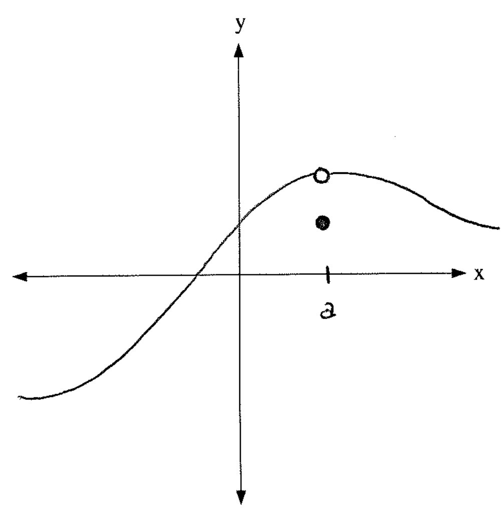
(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

(b) Draw graphs of two functions that are discontinuous in different ways. Explain in words how each one fails to fulfill your defintion in (a).





In the first graph, $\lim_{x \to a} f(x)$ does not exist. In the second graph, $\lim_{x \to a} f(x) \neq f(a)$.

5. Limits are the key to the definitions in calculus. State the **definition** of the derivative of a function $f(x)$.

The derivative of a function f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists and x is in the domain of f.

6. Using algebraic manipulation and short-cuts (rules rather than the definition of the derivative), differentiate the following:

(a) $f(x) = \sqrt[4]{x} + \dfrac{2}{\sqrt[4]{x}}$

$$f(x) = \sqrt[4]{x} + \frac{2}{\sqrt[4]{x}} = x^{\frac{1}{4}} + 2x^{-\frac{1}{4}}, \text{ so } f'(x) = \frac{1}{4}x^{-\frac{3}{4}} + 2 \cdot -\frac{1}{4}x^{-\frac{5}{4}} = \frac{1}{4x^{\frac{3}{4}}} + \frac{-1}{2x^{\frac{5}{4}}}$$

(b) $g(x) = x^9 \tan x$

$$g'(x) = x^9 \sec^2 x + 9x^8 \tan x$$

(c) $y = \dfrac{x^6 + 5x^3 - 7x}{x^4}$

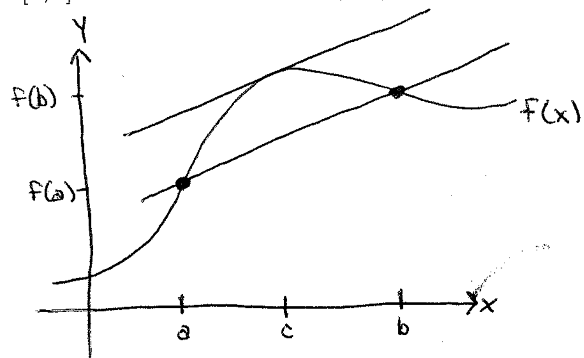
$$y = \frac{x^6 + 5x^3 - 7x}{x^4} = x^2 + \frac{5}{x} - \frac{7}{x^3} = x^2 + 5x^{-1} - 7x^{-3}, \text{ so } y' = 2x - 5x^{-2} + 21x^{-4} = 2x - \frac{5}{x^2} + \frac{21}{x^4}$$

(d) $y = \sec(\ln(8x^3))$

$$\frac{dy}{dx} = \sec(\ln(8x^3))\tan(\ln(8x^3)) \cdot \frac{1}{8x^3} \cdot 24x^2 = \frac{3\sec(\ln(8x^3))\tan(\ln(8x^3))}{x}$$

7. The Mean Value Theorem is key in proving the Fundamental Theorem of Calculus, which we will see later this semester. State the Mean Value Theorem completely. Then draw a picture that illustrates its meaning.

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a "c" in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



8. Suppose you are given the graph of g' and want to draw the graph of g. Given the information about g' below, what can you say about g? Write full sentences for your answers. Note when answering one part of this problem you may have to refer to other parts in order to give complete answers. Use the graphs in problem 8 and your knowledge of what derivatives tell you to help you think backwards.

(a) g' is horizontal on the interval $[a, b]$

g is a straight line on the interval $[a, b]$

(b) g' is negative on the interval (b, c)

g is decreasing on the interval $[b, c]$

(c) g' is positive on the interval (c, d)

g is increasing on the interval $[c, d]$

(d) $g'(c) = 0$

since g also changes from decreasing to increasing at c, g has a relative minimum at $x = c$

(e) $g'(b)$ is not defined

g has a discontinuity at $x = b$

(f) g' has a relative minimum or maximum at e

g has an inflection point at $x = e$

9. Evaluate $\lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)}$.

$$\lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)} = \left(\frac{0}{0}\right) \overset{H}{=} \lim_{x\to 0} \frac{e^x}{\frac{1}{1+x}\cdot 1} = \frac{1}{1} = 1$$

10. Evaluate the following indefinite integrals (remember this means "find the antiderivative of the integrand").

(a) $\int (\sec x \tan x + \sin x - \sqrt[5]{x})dx$

$$\int (\sec x \tan x + \sin x - \sqrt[5]{x})dx = \int \sec x \tan x dx + \int \sin x dx - \int x^{\frac{1}{5}} dx = \sec x - \cos x - \frac{5}{6}x^{\frac{6}{5}} + C$$

(b) $\int \frac{6x^4 - 7x^3 + 8x^2 - 9}{x^3} dx$

$$\int \frac{6x^4 - 7x^3 + 8x^2 - 9}{x^3} dx = \int \left(6x - 7 + \frac{8}{x} - \frac{9}{x^3}\right) dx = \int \left(6x - 7 + \frac{8}{x} - 9x^{-3}\right) dx$$

$$= 3x^2 - 7x + 8\ln|x| + \frac{9}{2x^2} + C$$