

Chapter 1 Theorems and Facts

MATH 204: Linear Algebra

THEOREMS:

Theorem 1: Uniqueness of Reduced Row-Echelon Form Theorem:

Each matrix is row equivalent to one and only one reduced echelon matrix.

Theorem 2: Existence and Uniqueness of Solutions Theorem:

A linear system is consistent if and only if the rightmost column of the augmented matrix is **not** a pivot column. That is, if and only if an echelon form of the matrix does not have a row of the form:

$$[0 \ 0 \ \cdots \ 0 \ b], \text{ with } b \neq 0.$$

If the linear system is consistent, then the solution set contains either (i) a unique solution when there are no free variables, or (ii) infinitely many solutions when there is at least one free variable.

Theorem 2.5: Properties of Scalar Multiplication and Vector Addition:

(see page 27 for the appropriate commutative, associative, distributive, etc. properties)

Theorem 3: Equivalent Representations Theorem:

Summary: $A\mathbf{x} = \mathbf{b}$ is the same as a vector equation and is solved using the augmented matrix $[A \ \mathbf{b}]$.

Corollary to Theorem 3:

$A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A (if and only if \mathbf{b} is in $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$). (Note that this becomes a part of the Connections Theorem later!)

Theorem 4: The Connections Theorem:

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

1. For each \mathbf{b} in \mathbb{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
3. The columns of A span \mathbb{R}^m .
4. A has a pivot position in every row.

Theorem 5: Properties (Linearity) of Matrix-Vector Multiplication Theorem:

see page 39

Theorem 6: Solution Sets of Nonhomogeneous Systems in Relation to Homogeneous Solutions Theorem:
see page 47

Theorem 7: Characterization of Linearly Dependent Sets Theorem:
see page 59

Theorem 8: Surplus of Vectors Theorem:
see page 60

Theorem 9: Dependence of Sets Containing the Zero Vector Theorem:
see page 60

FACTS:

Fact 1: Linear Systems with Row Equivalent Matrices Fact:

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Facts 2 a and b: One Pivot Per Row (Column):

Each row contains at most one pivot. (Each column contains at most one pivot.)

Fact 3: Nontrivial Homogeneous Solutions Fact:

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Fact 4: Independence of Matrix Columns Fact:

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Fact 5: Linear Independence of Sets of Two Vectors Fact:

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.