Characterizing a Subclass of Well-Covered Graphs

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Abstract

A graph G is said to be *well-covered* if every maximal independent set of G is of the same size. It has been shown that characterizing well-covered graphs is a co-NP-complete problem. In an effort to characterize some of these graphs the author has focused on well-dominated graphs, a class of graphs Finbow, Hartnell and Nowakowski have proved to be a subclass of the well-covered graphs. A set of vertices D is said to be a *dominating set* of a graph G if every vertex in G is either in D or adjacent to a vertex in D. A graph is *well-dominated* if every minimal dominating set is minimum. In an attempt to characterize well-dominated graphs that are 3-connected, planar, and claw-free, the author was able to extend the result to a characterization of all well-covered graphs with these three properties. In this paper we will describe this characterization and outline the arguments of the proof. The full details may be found at http://math.hws.edu/eking.

In this paper, assume any graph $G = (V, E)$ is a finite simple graph with vertex set V and edge set E. The notation $u \sim v$ denotes that vertices u and v are adjacent, while $u \nsim v$ denotes that they are not. The *independence number* of G, denoted $\alpha(G)$, is defined as the maximum cardinality of all independent sets of G. We say that a graph is *well-covered* if every maximal independent set of G has the same cardinality. Determining whether or not a graph is well-covered has been shown to be co-NP-complete [2] [7].

A *dominating set* $D \subseteq V$ of G is a set such that each vertex $v \in V$ is either in the set or adjacent to a vertex in the set. The *domination number* of G, denoted $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. Note that $\gamma(G) \leq \alpha(G)$ for all graphs G, since any maximal independent set is a minimal dominating set. A graph is *well-dominated* if every minimal dominating set of G has the same cardinality. The following, proved by Finbow, Hartnell and Nowakowski relates the properties of being well-covered and well-dominated.

Lemma 1 [4]: Every well-dominated graph is well-covered.

The author's original goal was to characterize all planar, 3-connected, claw-free, well-dominated graphs. Realizing that her arguments could be generalized, she characterized all planar, 3-connected, claw-free, well-covered graphs and concluded that nearly all of them are also well-dominated. Since the proof is too long to be printed here, this paper outlines the proof. The full details can be found at http://math.hws.edu/eking.

The following lemma, due to Campbell and Plummer [1], is an important tool in proving the characterization theorems.

Lemma 2 [1]: Let G be a well-covered graph and I be an independent set of G. If C is a component of $G - N[I]$, then C is well-covered.

The proof of the theorem characterizing planar, 3-connected, claw-free, well-covered graphs is broken down into subcases determined by the possible degree of a given vertex. The following theorem is vital in narrowing down possible subcases.

Theorem 3 [6]: If G is 3-connected, claw-free and planar, then
(a) $d(v) \leq 6$ for all $v \in V(G)$, and
(b) if v has degree 6 in G, then v lies on at least two separating triangles.

Given this theorem, there are only a finite number of possibilities for the degree of a vertex in a graph G that is 3-connected, claw-free, planar and well-covered. Considering all possibilities for the degree of a vertex v, we will explore the graph by proceeding from v and stopping regularly to check that all of the hypotheses still hold. Usually we will be considering only a portion of G (that portion to which we have "traveled" and can "see"). Let a *semi-known* subgraph S of G be an induced subgraph of G on the vertices for which we have complete adjacency information at a given time in the argument, union the other vertices and edges to which we have traveled but about which have only partial information. Vertices for which we have only partial adjacency information may be adjacent to other such vertices or to vertices to which we have not yet traveled. These vertices for which we have partial adjacency information are said to have the ability to *grow*, and a set of vertices can grow if any of the vertices in the set can grow.

The proof makes extensive use of Lemma 2. Often an independent set I of vertices in the semi-known subgraph S is chosen in such a way that there is a component of $S - N[I]$ that is not well-covered. By Lemma 2, there cannot be such a component remaining when we delete an independent set from a well-covered graph G, and thus there must be a vertex of S that can grow. The following lemma describes the different possibilities for the growth of S.

Lemma 4: Let G be a well-covered graph and S be a semi-known subgraph of G. Suppose that I is an independent set of vertices of S and C is a component of $S - N[I]$. If C is not well-covered, then either

(i) there is an edge, $e \notin E(S)$, in G between two vertices of I, i.e. I is dependent,

or (ii) there is an edge, $e \notin E(S)$, in G between two vertices of C,

or (iii) there is an edge, $e \notin E(S)$, in G between a vertex of I and a vertex of C,

or (iv) there is an edge, $e \notin E(S)$, in G between a vertex of C and a vertex of $V(G) - V(C) - I$ that is not adjacent to any vertex in I.

Proof: Assume the hypotheses of the lemma hold. Suppose by way of contradiction that C is not well-covered and none of (i), (ii), (iii), or (iv) hold. Since (i) does not hold, I is an independent set of G. Since (ii) and (iii) do not hold, C is a subgraph of $G - N[I]$. Since (iv) does not hold, C is not a proper subgraph of any larger connected subgraph of $G - N[I]$. Thus C is a component of $G - N[I]$. By Lemma 2, G is not well-covered, a contradiction. Therefore at least one of (i), (ii), (iii), or (iv) must hold. ■

The following term is used when the fourth possibility of Lemma 4 occurs. Let S be a semi-known subgraph of a graph G, I be a set of vertices in S that is independent in G, and C be a component of $S - N[I]$ that is not well-covered. Then any vertex v of $G - S$ that is adjacent to a vertex of C is said to be *born* by the deletion of $N[I]$. Thus we also say that v is not adjacent to any vertices of I *by birth*.

The following lemma will be very useful in eliminating possible subcases from the proof.

Lemma 5: Let G be a planar, claw-free, 3-connected graph, and S be a semi-known subgraph of G. Suppose there exists a set $I \subseteq V(S)$ that is independent in G, and a component, C, of $S - N[I]$ consisting of a vertex, v, and a subset of its neighbors such that: (i) v cannot grow, and (ii) there are at least two neighbors of v in C that cannot grow and are independent from one another. Then G is not well-covered.

Proof: Let S be a semi-known subgraph of G. Let I be a set of vertices such that $I \subseteq V(S)$ and I is independent in G, where $S - N[I]$ has a component C containing a vertex v, which cannot grow, and a subset of the neighbors of v. Let v_1 and v_2 be neighbors of v in C, such that $v_1 \nsim v_2$ and v_1 and v_2 cannot grow. Since v cannot grow, all the adjacencies of v in G are known and so $I \cup \{v\}$ is independent in G. Extend $I \cup \{v\}$ to a maximal independent set of G; call it J. Let $T = J - \{v\}$. Note that since C is a component of $S - N[I]$, there are no edges between the vertex set $\{v, v_1, v_2\}$ and vertices of T. Thus $T \cup \{v_1, v_2\}$ is also independent in G. If this set is maximal independent in G, call it J'; otherwise extend it to a maximal independent set J' in G. Then $|J'| \geq |J| + 1$ and so G is not well-covered. ■

Define $\mathcal{G}$ to be the class of graphs containing K_4, and those graphs formed by a collection of K_4's drawn in the plane and connected by edges joining exterior vertices of these K_4's in such a way that G is 3-connected, plane and has the following property: if an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$.

In Figures 6-8 where semi-known graphs are illustrated, a vertex for which only partial adjacency information is known is signified by showing dotted lines extending from the vertex to the unknown parts of G.

Theorem 6: Let G be a planar, 3-connected graph. Then G is claw-free and well-covered if and only if G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$.

Outline of Proof of Theorem 6:

Note that some claims are not proved here and others are only outlined. Please see the aforementioned website for full details.

Claim 6/1: If G is one of the exceptional graphs in Figure 1 or Figure 2 or G is in the class $\mathcal{G}$, then G is planar, 3-connected, claw-free and well-covered.

Proof of Claim 6/1: It is left to the reader to check that the graphs in Figures 1 and 2 are planar, 3-connected, claw-free and well-covered. Clearly K_4 is planar, 3-connected, claw-free and well-covered. So suppose that G is a graph the class $\mathcal{G}$ that is not K_4. By definition of $\mathcal{G}$, G is planar and 3-connected.

Suppose, by way of contradiction, that G contains a claw at v, a vertex of G, with vertices u, w and x. Then each of u, w and x are adjacent to v, but $\{u, w, x\}$ is an independent set. Since any two vertices within the same K_4 must be adjacent and there exists a claw at v, exactly two of $\{u, w, x\}$ must be neighbors of v from outside the K_4 to which v belongs, and these neighbors must be in distinct K_4's. Suppose u is the vertex in the K_4 to which v belongs, and w and x are neighbors in other K_4's. But then by definition of $\mathcal{G}$, if v is joined to two vertices w and x in two other K_4's, then $w \sim x$. Thus there is no claw at v. Hence G is claw-free.

Finally, we must show that G is well-covered. Clearly, any maximal independent set of G may contain at most one vertex from each K_4. By planarity and 3-connectivity, there is one vertex from each K_4 that is not connected to any other K_4's. Thus every maximal independent set of G must contain exactly one vertex from each K_4. Therefore every maximal independent set has the same cardinality, and therefore G is well-covered.

Thus if G is one of the exceptional graphs in Figure 1 or Figure 2 or G is in the class $\mathcal{G}$, then G is planar, 3-connected, claw-free and well-covered. ∎

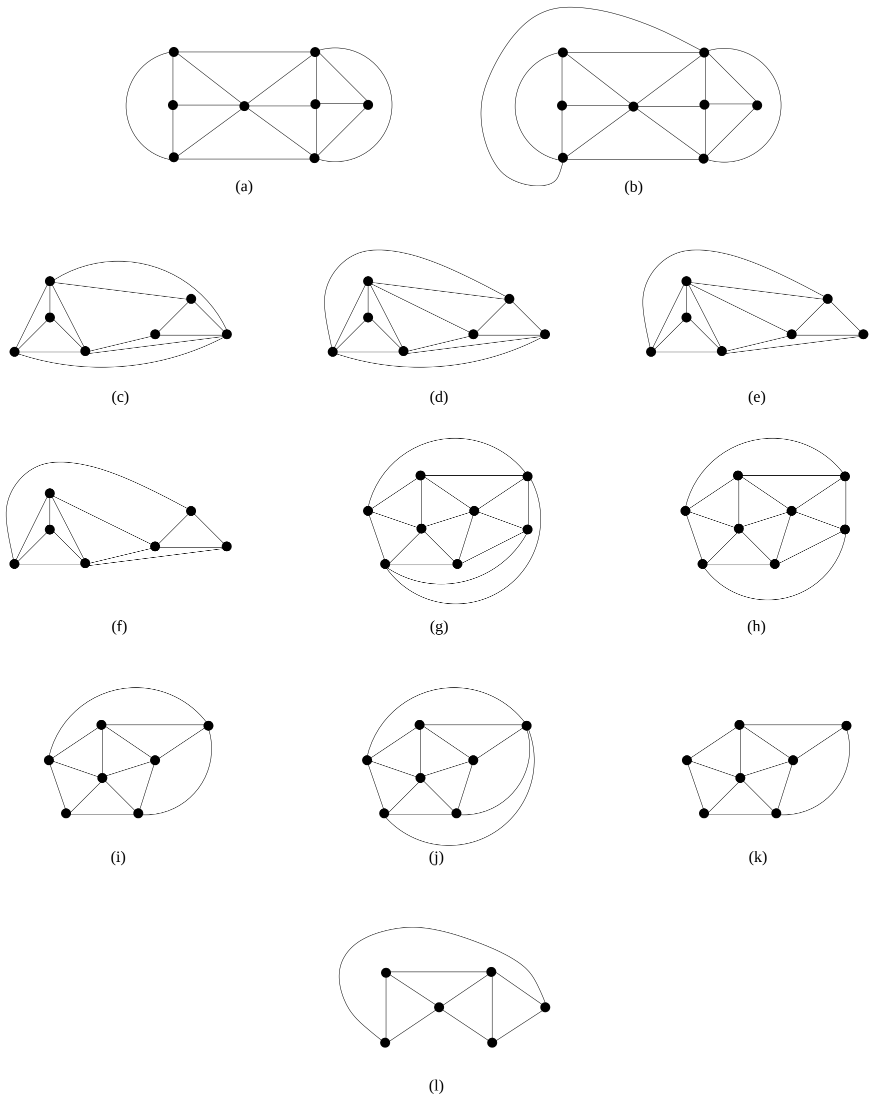

Figure 1: Exceptional well-covered, claw-free, planar, 3-connected graphs

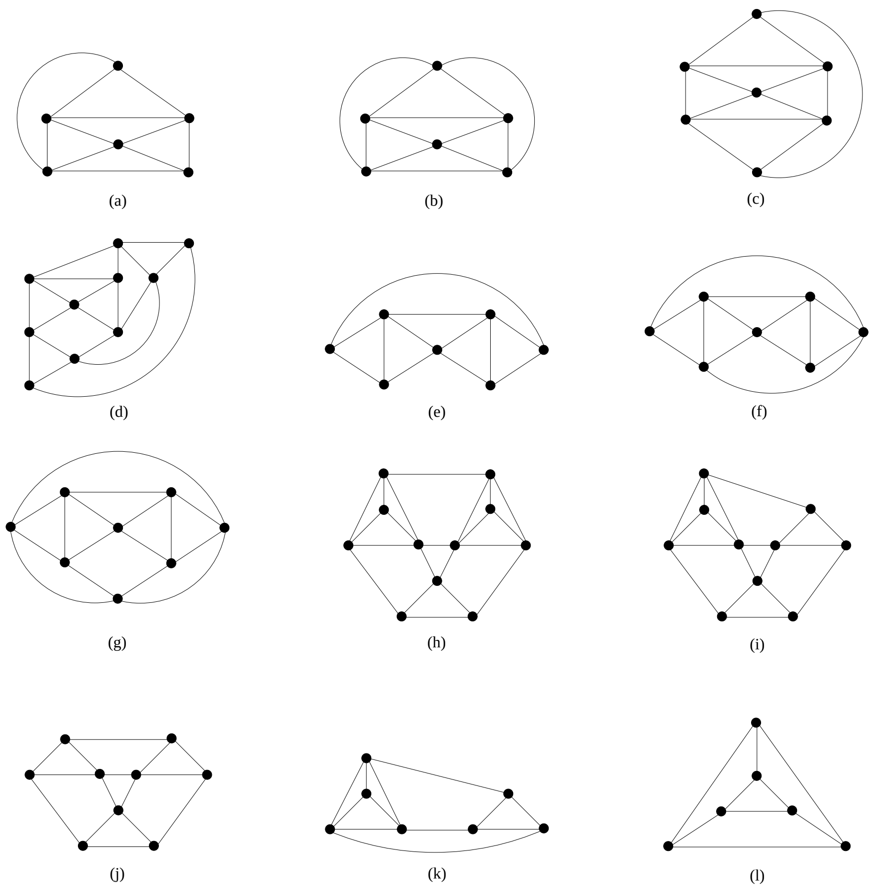

Figure 2: More exceptional well-covered, claw-free, planar, 3-connected graphs

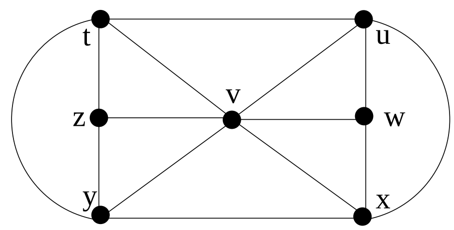

Figure 3: The local structure for a semi-known subgraph with $d(v) = 6$.

Claim 6/2: If G is planar, 3-connected, claw-free and well-covered, then G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$.

Outline of Proof of Claim 6/2: Suppose that G is planar, 3-connected, claw-free and well-covered.

Claim 6/2.1: If G has a vertex of degree six, then G is one of the first two exceptional graphs in Figure 1.

Outline of Proof of Claim 6/2.1: Let v be a vertex of G such that $d(v) = 6$. Label the neighbors of v in clockwise order: u, w, x, y, z, and t. By Theorem 3, v must lie on at least two separating triangles, and so without loss of generality, assume that $u \sim x$ and $y \sim t$. Then by claw-freedom, $w \sim u$, $w \sim x$, $z \sim y$ and $z \sim t$. Note that the induced subgraph S on $\{u, v, w, x, y, z, t\}$ is not well-covered, since $\{v\}$ and $\{z, w\}$ are both maximal independent sets of S. Therefore, S must not be all of G and at least one vertex of S must grow. By 3-connectivity, the four triangular faces having v as a corner vertex contain no additional vertices in G.

Claim 6/2.1.1: There must be at least two disjoint edges between the vertex sets $\{u, x\}$, and $\{t, y\}$, and there are no vertices in the exterior face, that is, the $uxyt$-face.

By Claim 6/2.1.1, we may assume we have the semi-known subgraph S shown in Figure 3. Note that this graph is not well-covered. The only additional edges that we can have between known vertices are uy or xt. By planarity, we cannot have both of these edges, so without loss of generality, suppose the only possible additional edge between known vertices is uy. Even if we add this edge, the resulting graph is not well-covered. Thus a vertex of S must grow into either the uwx-face or the yzt-face. Without loss of generality, suppose either u, w or x is adjacent to a new vertex in the uwx-face. By 3-connectivity, if one of u, w or x is adjacent to an additional vertex in the uwx-face then each of u, w and x is adjacent to an additional vertex in the uwx-face. Hence u is adjacent to an additional vertex in the uwx-face; call it s. To prevent $\{t, w, s\}$ and $\{t, x, s\}$ from forming claws with u, we must have that $w \sim s$ and $x \sim s$. This graph, with or without the additional edge uy is well-covered, and thus we have the first two exceptional graphs shown in Figure 1.

The following Claim shows that G cannot contain any additional vertices and so must be one of these two graphs.

Claim 6/2.1.2: The graph G must contain exactly eight vertices.

Outline of Proof of Claim 6/2.1.2: Above it was shown that G cannot contain fewer than eight vertices. We now argue that no more vertices may be added to the first two graphs in Figure 1 to obtain larger well-covered graphs containing a vertex of degree six. Let S be a semi-known subgraph of G and one of the first two graphs in Figure 1. Suppose, by way of contradiction, we can add vertices to S to form a larger planar, 3-connected, claw-free, well-covered graph. By claw-freedom, there are no additional vertices in the usx-face; otherwise by 3-connectivity u would have an additional neighbor in this face and there would be a claw at u with t, w and the additional neighbor of u. Recall by Claim 6/2.1.1, there are no vertices in the exterior face. Thus, any additional vertices must be in either the yzt-face, the uws-face, or in the xws-face. Note that by 3-connectivity, if there exists an additional vertex in one of these faces all three vertices that make up the corners of this face must have a neighbor inside the face. Thus there cannot be additional vertices in both the uws-face and the xws-face, or v together with an additional neighbor of w from each of these faces forms a claw at w. If the edge uy is not in our graph, the uws-face is symmetric to the xws-face. If the edge uy is in our graph, then u already has six neighbors and so by Theorem 3 cannot have any neighbors in the uws-face. Thus without loss of generality, we may suppose that any additional vertex must be either in the yzt-face or in the xws-face, and no additional neighbors are in the uws-face.

First it is proven that no additional vertex may be in the xws-face. This is shown by proving that such vertices would need to be in the form of *345-nests* described by Plummer [6], and then by showing that if the graph has such a configuration in the xws-face then it is not well-covered. Then it is proven that no additional vertices may be in the yzt-face.

Hence the graph G, containing a vertex of degree six, must have exactly eight vertices. ■

Therefore if G has a vertex of degree six, then G is one of the first two exceptional graphs in Figure 1. ■

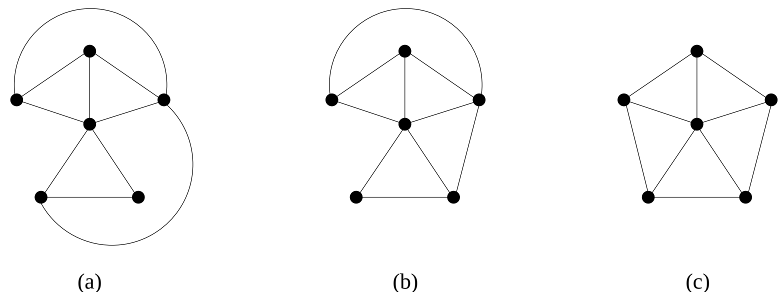

Figure 4: Three forbidden subgraphs containing a vertex of degree five.

The following claim will be useful.

Claim 6/2.2: Let G be a 3-connected, claw-free, plane graph, with the property that $d(v) \leq 5$ for all v in G. Suppose v is a vertex of G that is interior to a 3-cycle, where v is adjacent to all three of the vertices on the boundary of this cycle, and one of the boundary vertices has degree five (or degree four and cannot grow) such that two (or one) of its neighbors are not adjacent to v and the other boundary vertices. Then v cannot grow and $d(v) = 3$.

Proof of Claim 6/2.2: Suppose G and v fulfill the hypotheses of the claim. Let u, w and x be the boundary vertices of the cycle, such that each of u, w and x is adjacent to v, and without loss of generality, suppose $d(u) = 5$ (or $d(u) = 4$ and u cannot grow). Then the two neighbors (or one neighbor) of u that are not adjacent to v or the other boundary vertices are exterior to the uwx-cycle by 3-connectivity. There can be no additional vertices in either the uvw-face or the uvx-face; otherwise $\{v, w\}$ or $\{v, x\}$ respectively would be 2-cuts, separating the additional vertices from the rest of the graph and contradicting the fact that G is 3-connected. Suppose there is an additional vertex in the vwx-cycle. Then by 3-connectivity, each of v, w and x must have a neighbor interior to this cycle. Thus w cannot have an additional neighbor exterior to the uwx-cycle; otherwise this additional exterior neighbor together with u and the additional neighbor in the vwx-cycle would be a claw at w, contradicting the fact that G is claw-free. But then $\{u, x\}$ is a 2-cut, separating v and w from the rest of the graph and contradicting the fact that G is 3-connected. Hence there is no vertex in the vwx-face either. Therefore v cannot grow and has $d(v) = 3$. ■

The remainder of the proof has several kinds of arguments that are repeatedly used with slight modifications. The author will outline the proof

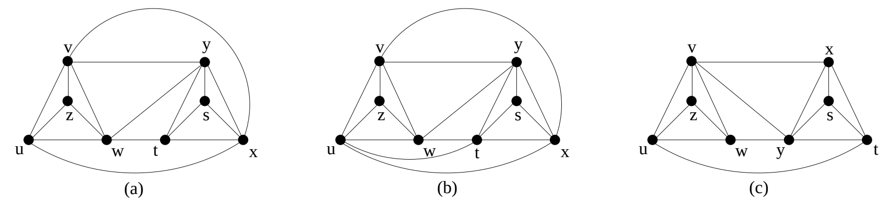

Figure 5: The only members of $\mathcal{G}$ containing a forbidden subgraph shown in Figure 4(b).

and give the reader an in-depth look at one particular subcase that highlights the main types of arguments. Again, the reader may find the full details at http://math.hws.edu/eking.

Claim 6/2.3: If G is not one of the exceptional graphs in Figure 1 or Figure 2, and G is not one of the graphs of $\mathcal{G}$ in Figure 5, then the graphs shown in Figure 4 are forbidden subgraphs of the graph G.

Claim 6/2.4: If G is not one of the exceptional graphs in Figure 1 or Figure 2, then every vertex of G must lie on a K_4.

Outline of Proof of Claim 6/2.4: Suppose that G is not one of the exceptional graphs in Figure 1 or Figure 2. Let v be a vertex of G. By Theorem 3 and Claim 6/2.1, $d(v) < 6$. Since G is 3-connected, $d(v) \geq 3$. Thus we must show that if $3 \leq d(v) \leq 5$, then v lies on a K_4.

Claim 6/2.4.1: If G is not one of the exceptional graphs in Figure 1 and v is a vertex of G with $d(v) = 5$, then v must lie on a K_4.

Claim 6/2.4.2: If G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 4$, then v must lie on a K_4.

Outline of Proof of Claim 6/2.4.2: Let G be a graph and v be a vertex of G that fulfill the hypotheses of the claim. Label the neighbors of v in a clockwise fashion: u, x, y, w. Suppose, by way of contradiction, that v does not lie on a K_4. By claw-freedom, and without loss of generality, we may assume that $x \sim y$ and $u \sim w$. We call this subgraph, the *bow-tie subgraph* centered at v. Since G is 3-connected, every vertex of G has degree at least three. Thus x must grow. Either x is adjacent to u, x is adjacent to w or x is adjacent to an additional vertex.

Claim 6/2.4.2.1: The vertex x is not adjacent to u.

Outline of Proof of Claim 6/2.4.2.1: Suppose, by way of contradiction,

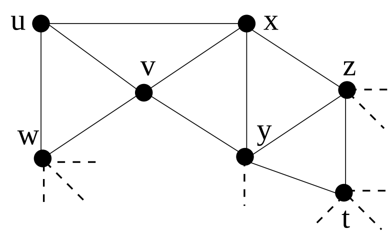

Figure 6: Proving that every vertex of degree four must lie on a K_4.

that $x \sim u$. Then $x \nsim w$ and $u \nsim y$; otherwise v would lie on a K_4. Thus either x is adjacent to an additional vertex or $d(x) = 3$. Suppose that x is adjacent to an additional vertex; call it z. To prevent $\{xu, xy, xz\}$ from forming a claw at x, either $z \sim u$ or $z \sim y$.

Claim 6/2.4.2.1.1: The vertex z is not adjacent to u.

Claim 6/2.4.2.1.2: The vertex z is not adjacent to y.

Outline of Proof of Claim 6/2.4.2.1.2: Suppose, by way of contradiction, that $z \sim y$. Either x is adjacent to an additional vertex or $d(x) = 4$.

Suppose x is adjacent to an additional vertex; call it t. To prevent $\{xv, xt, xz\}$ from forming a claw at x, we must have $t \sim z$. To prevent $\{xt, xu, xy\}$ from forming a claw at x, either t is adjacent to u or t is adjacent to y. Recall $u \nsim y$ since v does not lie on a K_4. Suppose $t \sim u$. Then t must be exterior to the xyz-face, and we have the forbidden 5-wheel shown in Figure 4(c) centered at x. Thus $t \nsim u$. Suppose $t \sim y$. Then we have the forbidden subgraph shown in Figure 4(b) centered at x. Thus $t \nsim y$, and therefore x is not adjacent to an additional vertex.

Suppose $d(x) = 4$. Then either u is adjacent to an additional vertex or $d(u) = 3$.

Claim 6/2.4.2.1.2.1: The vertex u is not adjacent to an additional vertex.

Claim 6/2.4.2.1.2.2: The vertex u must be adjacent to an additional vertex.

Proof of Claim 6/2.4.2.1.2.2: Suppose, by way of contradiction, that u is not adjacent to an additional vertex and so $d(u) = 3$. Either y is adjacent to an additional vertex, or $d(y) = 3$.

Suppose y is adjacent to an additional vertex; call it t. To prevent $\{yv, yz, yt\}$ from forming a claw at y, we must have $z \sim t$. (See Figure 6 for an illustration.) Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 4, either w is adjacent to t, or w is adjacent to an additional vertex. (Note that $w \nsim x$ since we are assuming that v is not in a K_4 and $d(x) = 4$.)

Suppose $w \sim t$. Then this semi-known subgraph of G is isomorphic to the exceptional well-covered graph shown in Figure 2(e). Since G is not a graph from Figure 2, this subgraph must grow. If $w \sim y$, then we have the forbidden subgraph shown in Figure 4(c) centered at y. Thus we may assume $w \nsim y$. Suppose y is adjacent to an additional vertex; call it s. Then to prevent $\{yv, yz, ys\}$ and $\{yv, yt, ys\}$ from forming claws at y, we must have $z \sim s$ and $t \sim s$. Thus s is in the zyt-face by planarity. But then we have the forbidden subgraph shown in Figure 4(b) centered at y. Thus y is not adjacent to an additional vertex. Hence since $w \nsim y$ and y is not adjacent to an additional vertex, y cannot grow and $d(y) = 4$. Note that if z is adjacent to an additional vertex, then t must also be adjacent to that vertex by claw-freedom at z. Thus, since the graph must grow, either w is adjacent to z, or t is adjacent to an additional vertex.

Suppose $w \sim z$. Then this semi-known subgraph of G is isomorphic to the exceptional well-covered graph shown in Figure 2(c). Since G is not a graph from Figure 2, this subgraph must grow. Since there are no additional edges between known vertices, w, z or t must be adjacent to an additional vertex. Note that by claw-freedom at w, if w is adjacent to an additional vertex, then t must be adjacent to that vertex as well. Similarly, if z is adjacent to an additional vertex, then t must be adjacent to that vertex as well. Hence since the graph must grow, t must be adjacent to an additional vertex; call it s. To prevent $\{ty, ts, tw\}$ from forming a claw at t, we must have $s \sim w$. To prevent $\{wu, wz, ws\}$ from forming a claw at w, we must have $s \sim z$. But then we have the forbidden subgraph shown in Figure 4(b) centered at z. Thus $w \nsim z$.

Suppose t is adjacent to an additional vertex; call it s. To prevent $\{ty, ts, tw\}$ from forming a claw at t, we must have $s \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, u and y cannot grow, and $u \nsim y$. Thus by Lemma 5, G is not well-covered, a contradiction. Thus t is not adjacent to an additional vertex, and therefore, $w \nsim t$.

Suppose w is adjacent to an additional vertex; call it s. (See Figure 7(a) for an illustration.) Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $s \nsim t$ by birth, $s \nsim y$ by claw-freedom at y. Thus by Lemma 4, either w is adjacent to y, w is adjacent to an additional vertex,

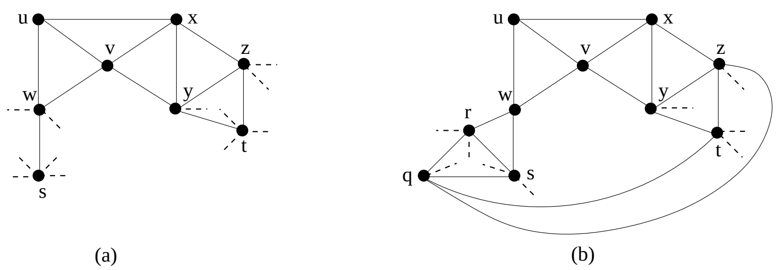

Figure 7: Proving that every vertex of degree four must lie on a K_4.

or s is adjacent to an additional vertex.

Suppose $w \sim y$. But then $\{yt, yw, yx\}$ is a claw at y since x cannot grow and $w \nsim t$ by the preceding case, which contradicts the fact that G is claw-free. Hence $w \nsim y$.

Suppose w is adjacent to an additional vertex; call it r. To prevent $\{wv, ws, wr\}$ from forming a claw at w, we must have $s \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s, r\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 4, either w is adjacent to an additional vertex, r is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it q. To prevent $\{wv, ws, wq\}$ and $\{wv, wr, wq\}$ from forming claws at w, we must have $s \sim q$ and $r \sim q$. Now $d(w) = 5$, and either q is interior to the wrs-face, r is interior to the wsq-face, or s is interior to the wrq-face. By Claim 6/2.2, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s, r, q\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u, and one of s, r and q, cannot grow, and u is not adjacent to any of s, r, or q. Thus by Lemma 5, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (as a fifth neighbor), and thus there are no additional vertices in the wrs-face by 3-connectivity.

Suppose r is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $y \nsim q$ by birth, and since $s \nsim t$ by birth, $s \nsim y$ by claw-freedom at y. Also since $w \nsim t$ by a previous subcase (of the y adjacent to an additional vertex subcase, a subcase among the subcases of Claim 6/2.4.2.1.2.2), $w \nsim y$ by claw-freedom at y. Thus by Lemma 4, either q is adjacent to s, or s is adjacent to an additional vertex.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 4, q must be adjacent to t. Since G is 3-connected, $\{w, t\}$ is not a 2-cut, and so there must be a path from y and z to the vertex set $\{q, r, s\}$ that does not pass through w or t. Note that r and q are not adjacent to y by birth, and since $s \nsim t$ by birth, $s \nsim z$ by claw-freedom at z. Thus either r is adjacent to z, q is adjacent to z, or z is adjacent to an additional vertex in the $ztqrwux$-face. Suppose $r \sim z$. To prevent $\{zx, zt, zr\}$ from forming a claw at z, we must have $r \sim t$. But then $\{rw, rq, rz\}$ is a claw at r, since w cannot grow and $q \nsim z$ by planarity. Thus $r \nsim z$. Suppose $q \sim z$. (See Figure 7(b) for an illustration.) Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, w and x cannot grow, and $w \nsim x$. Thus by Lemma 5, G is not well-covered, a contradiction. Hence $q \nsim z$. Suppose z is adjacent to an additional vertex in the $ztqrwux$-face; call it p. To prevent $\{zx, zt, zp\}$ from forming a claw at z, we must have $t \sim p$. To prevent $\{tp, ty, tq\}$ from forming a claw at t, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim s$ by planarity. Thus by Lemma 4, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Hence n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 4(b) centered at y. Thus z is not adjacent to an additional vertex, and hence $q \nsim s$.

Suppose s is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, t]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth, and C cannot grow. Thus by Lemma 4, either q is adjacent to t or p is adjacent to t.

Suppose $q \sim t$. (See Figure 8(a) for an illustration.) Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall $q \nsim s$ by the preceding subcase. Thus by Lemma 4, either q is adjacent to z, y is adjacent to an additional vertex, or z is adjacent to an additional vertex. Suppose $q \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, w and x cannot grow, and $w \nsim x$. Thus by Lemma 5, G is not well-covered, a contradiction. Hence $q \nsim z$. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$

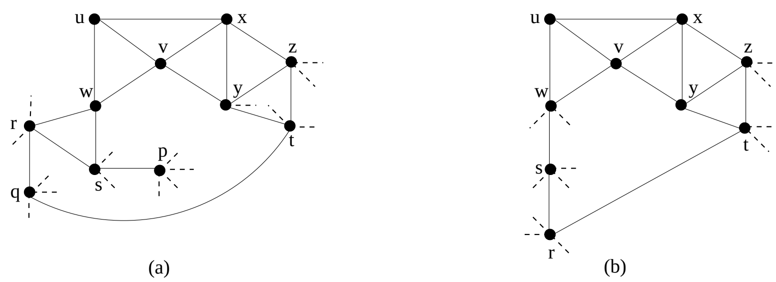

Figure 8: Proving that every vertex of degree four must lie on a K_4.

and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Thus n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 4(b) centered at y. Thus y is not adjacent to an additional vertex, and by 3-connectivity there are no additional vertices in the yzt-face. Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zx, zt, zn\}$ from forming a claw at z, we must have $t \sim n$. But then $\{tn, ty, tq\}$ is a claw at t, since $n \nsim q$ by birth. This contradicts the fact that G is claw-free, and hence z is not adjacent to an additional vertex, so $q \nsim t$.

Suppose $p \sim t$. Note that if y is adjacent to an additional vertex; call it n, then to prevent $\{yv, yz, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Thus n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 4(b) centered at y. Thus y is not adjacent to an additional vertex, and by 3-connectivity, there are no additional vertices in the yzt-face. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then all the vertices of $C - x$ are adjacent to x, vertices x, u and y cannot grow, and $u \nsim y$. Note that since $p \nsim q$ by birth, $p \nsim r$ by claw-freedom at r. Thus by Lemma 5, G is not well-covered, a contradiction. Hence $p \nsim t$, and so s is not adjacent to an additional vertex (when r is adjacent to an additional vertex), and therefore r is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $q \nsim y$ by birth, and C cannot grow. Thus by Lemma 2, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex (when w is adjacent to an additional vertex), and so w is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[r, t]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 4, r is adjacent to t. (See Figure 8(b) for an illustration.) Suppose y is adjacent to an additional vertex; call it q. Then to prevent $\{yv, yz, yq\}$ and $\{yv, yt, yq\}$ from forming claws at y, we must have $z \sim q$ and $t \sim q$. Thus q must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 4(b) centered at y. Thus y is not adjacent to an additional vertex, $d(y) = 4$, and by 3-connectivity, there are no additional vertices in the yzt-face. Since G is 3-connected, $\{w, t\}$ is not a 2-cut, and so there must be a path from z to s and r that does not pass through either w or t. Note that since $s \nsim t$ by birth, $s \nsim z$ by claw-freedom at z. Thus either r is adjacent to z, or z is adjacent to an additional vertex.

Suppose $r \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2, G is not well-covered, a contradiction. Hence $r \nsim z$.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zx, zt, zq\}$ from forming a claw at z, we must have $t \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 4, we must have $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[s, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 4, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so s is not adjacent to an additional vertex. Therefore w is not adjacent to an additional vertex, and finally y is not adjacent to an additional vertex, which means $d(y) = 3$.

Suppose $d(y) = 3$. Since G is 3-connected and $d(w) = 2$, w must grow. The only possible additional edge between w and other known vertices is wz. Suppose w is adjacent to z. Then this semi-known subgraph of G is isomorphic to the exceptional well-covered graph shown in Figure 1(l). Since G is not a graph from Figure 1, this subgraph must grow. There are no possible additional edges between known vertices, and so z or w must be adjacent to an additional vertex. But then $\{w, z\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence $w \nsim z$. Then w must be adjacent to an additional vertex, but then $\{w, z\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence $d(y) \neq 3$. But this is a contradiction, since above we showed that y must have degree three. Therefore $d(u) \neq 3$, u must be adjacent to an additional vertex, and we have proved Claim 6/2.4.2.1.2.2.■

By Claim 6/2.4.2.1.2.2, u must be adjacent to an additional vertex, but by Claim 6/2.4.2.1.2.1 u must not be adjacent to an additional vertex. Thus we have a contradiction and so we have proved Claim 6/2.4.2.1.2: the vertex z is not adjacent to the vertex y. Claim 6/2.4.2.1.1 proves that $z \nsim u$. But then $\{xu, xy, xz\}$ is a claw at x, contradicting the fact that G is claw-free. (Recall that $u \nsim y$ since v is not contained in a K_4.) Hence x is not adjacent to an additional vertex and $d(x) = 3$. By symmetry, we may also assume that $d(u) = 3$. The graph is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of the graph, thus the graph must grow. If either w or y is adjacent to an additional vertex, then $\{w, y\}$ is a 2-cut, separating u, v, and x from the rest of the graph, and contradicting the fact that G is 3-connected. Thus there are no additional vertices. But the only possible additional edge is wy, leaving us with a 4-wheel which is not well-covered. Hence $x \nsim u$, and we have proved Claim 6/2.4.2.1. ■

Similar arguments prove the following:

Claim 6/2.4.2.2: The vertex x is not adjacent to w.

Claim 6/2.4.2.3: The vertex x is not adjacent to an additional vertex.

Thus every vertex of degree four must lie on a K_4. ■

Claim 6/2.4.3: If G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 3$, then v must lie on a K_4.

By Theorem 3 and Claim 6/2.1, $d(v) < 6$. Hence since G is 3-connected, we know that $3 \leq d(v) \leq 5$ for all v in $V(G)$. By Claim 6/2.4.1 we know that every vertex of degree five must lie on a K_4. By Claim 6/2.4.2 we know that every vertex of degree four must lie on a K_4. By Claim 6/2.4.3 we know that every vertex of degree three must lie on a K_4. Therefore we have shown that if G is not one of the exceptional graphs in Figure 1 or Figure 2, then every vertex of G must lie on a K_4, and so we have proved Claim 6/2.4. ■

Claim 6/2.5: If G is not one of the exceptional graphs in Figure 1 or Figure 2, then any two K_4's in G must be disjoint.

Proof of Claim 6/2.5: Let G be a graph that fulfills the hypothesis of the claim. Suppose, by way of contradiction, that there exist two K_4's, $K_4(1)$ and $K_4(2)$, in G that are not disjoint. By planarity, these two K_4's may share at most three vertices. Recall that by Theorem 3, $d(v) \leq 6$ for all v in $V(G)$. Suppose $K_4(1)$ and $K_4(2)$ share exactly one vertex, v. Then

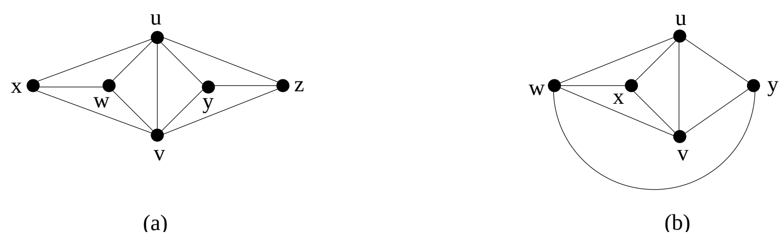

Figure 9: Proving that any two K_4's of G are disjoint.

v must be adjacent to three vertices of $K_4(1)$ and three vertices of $K_4(2)$ that are all distinct, since $K_4(1)$ and $K_4(2)$ share exactly one vertex. But then $d(v) = 6$, and so by Claim 6/2.1, G must be the exceptional graph in Figure 1(a) or Figure 1(b). This contradicts the fact that G is not a graph in Figure 1, and so $K_4(1)$ and $K_4(2)$ cannot share exactly one vertex.

Suppose $K_4(1)$ and $K_4(2)$ share exactly two vertices, v and u. Since v and u are in a K_4 together, we know that $v \sim u$. Let w and x be the remaining two vertices of $K_4(1)$, such that w is interior to the uvx-face. Let y and z be the remaining two vertices of $K_4(2)$, such that y is interior to the uvz-face. (See Figure 9(a) for an illustration.) Now $d(u) = d(v) = 5$ and so by Claim 6/2.2, neither w nor y may grow. Since G is 3-connected, $\{x, z\}$ is not a 2-cut. Therefore G cannot contain any additional vertices, and we must have $x \sim z$. But then G is not well-covered, since $\{u\}$ and $\{w, y\}$ are both maximal independent sets of G. Hence $K_4(1)$ and $K_4(2)$ cannot share exactly two vertices.

Suppose $K_4(1)$ and $K_4(2)$ share exactly three vertices, v, u and w. Let x be the fourth vertex of $K_4(1)$ and y be the fourth vertex of $K_4(2)$. (See Figure 9(b) for an illustration.) Note that this semi-known subgraph of G is not well-covered, since $\{u\}$ and $\{x, y\}$ are both maximal independent, and so this subgraph must grow. Since G is 3-connected, $\{w, y\}$ is not a 2-cut, and so either u or v must be adjacent to an additional vertex. Without loss of generality, suppose that u is adjacent to an additional vertex; call it z. To prevent $\{ux, uy, uz\}$ from forming a claw at u, we must have $y \sim z$. Now $d(u) = 5$, and so by Claim 6/2.2, x cannot grow and $d(x) = 3$. Since G is 3-connected, $\{u, y\}$ is not a 2-cut and so there must be a path from z to w that does not pass through either u or y. Thus either w is adjacent to z, or w is adjacent to an additional vertex in the exterior face.

Suppose $w \sim z$. Then $d(w) = 5$, and so by Theorem 3 and Claim 6/2.1, w cannot grow. Thus there are no additional vertices in the vwy-face; otherwise $\{v, y\}$ would be a 2-cut, contradicting the fact that G is 3-connected. Furthermore, there are no additional vertices in the wyz-face; otherwise $\{x, z\}$ would be a 2-cut, contradicting the fact that G is

3-connected. Hence this semi-known subgraph of G cannot grow. But it is not well-covered, since $\{u\}$ and $\{x, z\}$ are both maximal independent sets of the graph. Therefore it cannot be a semi-known subgraph of G and $w \nsim z$.

Suppose w is adjacent to an additional vertex in the exterior face; call it t. To prevent $\{wx, wy, wt\}$ from forming a claw at w, we must have $t \sim y$. To prevent $\{yv, yz, yt\}$ from forming a claw at y, we must have $t \sim z$. Note that $d(w) = d(y) = 5$, and so by Theorem 3 and Claim 6/2.1, w and y cannot grow. By Claim 6/2.4, z must lie on a K_4. Since two of z's neighbors (u and y) cannot grow, z must then be adjacent to an additional vertex; call it s. To prevent $\{zu, zt, zs\}$ from forming a claw at z, we must have $t \sim s$. But then $\{z, t\}$ is a 2-cut, separating s from the rest of the graph and contradicting the fact that G is 3-connected. Therefore w is not adjacent to an additional vertex.

Hence $K_4(1)$ and $K_4(2)$ cannot share exactly three vertices.

Therefore $K_4(1)$ and $K_4(2)$ share none of their vertices; that is any two K_4's in G must be disjoint. ■

Claim 6/2.6: If an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$.

Proof of Claim 6/2.6: Let v be an exterior vertex of a K_4 in G that is adjacent to two vertices u and w on two other K_4's. Suppose, by way of contradiction, that $u \nsim w$. Let x be the interior vertex of v's K_4. Then $\{vx, vu, vw\}$ is a claw at v, contradicting the fact that G is claw-free. Thus if an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$. ■

If G is planar, 3-connected, claw-free and well-covered, by Theorem 3, $d(v) \leq 6$ for all v in $V(G)$. By Claim 6/2.1, if G has a vertex of degree six, G is one of two graphs in Figure 1. By Claim 6/2.4, if G is not a graph in Figure 1 or Figure 2, then every vertex of G must lie on a K_4. By Claim 6/2.5, these K_4's must be distinct. By Claim 6/2.6, if an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$. Thus if G is planar, 3-connected, claw-free and well-covered, then G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$. Hence we have proved Claim 6/2. ■

Corollary 7: Let G be a planar, 3-connected graph. Then G is claw-free and well-dominated if and only if G is one of the exceptional graphs in Figure 1 or Figure 2(a)-(j), or G is in the class $\mathcal{G}$.

Proof: Let G be a planar, 3-connected graph. By Theorem 6, G is claw-free and well-covered if and only if G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$. By Lemma 1, the set of claw-free, well-dominated graphs is a subset of the claw-free, well-covered graphs. Thus we must only determine which of the exceptional graphs in Figure 1, Figure 2, and the class $\mathcal{G}$ are also well-dominated.

We leave it to the reader to check that the all of the exceptional graphs in Figure 1, and the exceptional graphs in (a)-(j) of Figure 2 are well-dominated. Note that the graph in Figure 2(k) is not well-dominated. This graph is a K_4 and a K_3 with the exterior vertices of the K_4 joined to the K_3 by a matching. We may minimally dominate this graph with two vertices by choosing one vertex from the K_4 and one from the K_3. We may also minimally dominate this graph by choosing all three of the exterior vertices of the K_4. Thus it is not well-dominated. Also note that the graph in Figure 2(l) is not well-dominated. This graph is formed by two K_3's joined by a matching. We may minimally dominate this graph with two vertices by choosing one vertex from each of the K_3's. We may also minimally dominate this graph by choosing all three of the vertices from one of the K_3's. Thus it is not well-dominated.

Finally we must show that all of the graphs in the class $\mathcal{G}$ are well-dominated. Suppose G is in the class $\mathcal{G}$. Clearly if a set of vertices of G contains one vertex from each K_4, then it is dominating. By planarity and 3-connectivity, there is one vertex from each K_4 that is not connected to any other K_4's. Thus every minimal dominating set of G must contain exactly one vertex from each K_4. Therefore every minimal dominating set has the same cardinality, and therefore G is well-dominated.

Therefore, G is claw-free and well-dominated if and only if G is one of the exceptional graphs in Figure 1 or Figure 2(a)-(j) , or G is in the class $\mathcal{G}$. ■

Note that graphs in the class $\mathcal{G}$ must have the properties described in the following two lemmas.

Lemma 8: Let G be a graph in the class $\mathcal{G}$ containing at least two K_4's. Then each of the three exterior vertices of each K_4 of G must be adjacent to one vertex of at least one other K_4.

Proof: Suppose $\{u, v, w\}$ are the exterior vertices of a K_4 of G, and x is the interior vertex. By way of contradiction, suppose that v is not adjacent to any vertices other than the set $\{u, w, x\}$. Then $\{u, w\}$ is a 2-cut, separating v and x from the rest of the graph and contradicting the fact that G is 3-connected. Hence v must be adjacent to a vertex of at least one other K_4. ■

Lemma 9: Let G be a graph in the class $\mathcal{G}$ containing at least three K_4's. Then no two vertices from the same K_4 may be adjacent to the same vertex in another K_4.

Proof: Suppose G fulfills the hypotheses of the lemma. Suppose $\{u, v, w\}$ are the exterior vertices of a K_4 of G, and x is the interior vertex. By way of contradiction, suppose that u and w are both adjacent to y, a vertex in another K_4. Suppose $\{y, z, t\}$ are the exterior vertices of y's K_4, and s is the interior vertex. Then $d(y) = 5$ and by Theorem 3 and Claim 6/2.1, y cannot grow. By definition of $\mathcal{G}$, s cannot grow and $d(s) = 3$. Since G is 3-connected, $\{v, y\}$ is not a 2-cut, and so u or w must grow. Without loss of generality, suppose that w grows. The vertex w cannot be adjacent to an additional vertex, or this additional vertex together with x and y will form a claw at w, contradicting the fact that G is claw-free. Thus w must be adjacent to either t or z. Without loss of generality, suppose that $w \sim t$. Now $d(w) = 5$, and so by Theorem 3 and Claim 6/2.1, w cannot grow. By definition of $\mathcal{G}$, x cannot grow. Since G is 3-connected, $\{v, z\}$ is not a 2-cut, and so either u or t must grow. Without loss of generality, suppose u grows. The vertex u cannot be adjacent to an additional vertex, or this additional vertex, together with x and y will form a claw at w, contradicting the fact that G is claw-free. Thus either u is adjacent to t or u is adjacent to z.

Suppose $u \sim t$. Then $d(u) = d(t) = 5$, and so by Theorem 3 and Claim 6/2.1, neither u nor t may grow. But then $\{y, t\}$ is a 2-cut, separating z and s from the rest of the graph and contradicting the fact that G is well-covered.

Suppose $u \sim z$. Then $d(u) = 5$, and so by Theorem 3 and Claim 6/2.1, u cannot grow. Now t cannot be adjacent to an additional vertex, since this additional vertex, together with w and s, would form a claw at t, contradicting the fact that G is claw-free. Also z cannot be adjacent to an additional vertex; otherwise this additional neighbor, together with u and s would form a claw at z, contradicting the fact that G is claw-free. Thus since G contains at least three K_4's and is connected, v must be adjacent to an additional vertex. But then v is a cut-vertex, contradicting the fact that G is 3-connected. Hence $u \nsim z$.

Therefore, no two vertices from the same K_4 may be adjacent to the same vertex in another K_4. ■

There are an infinite number of planar, 3-connected, claw-free, well-covered (and well-dominated) graphs. To show that this is true, we will discuss several techniques for enlarging a graph of $\mathcal{G}$ with n K_4's to a graph of $\mathcal{G}$ with $n + 3$ or $n + 1$ K_4's. To prove that the resulting graphs are 3-connected, we will need the following terminology and theorem, which is an extension of Menger's Theorem.

Definition [8]: Given a vertex x and a set U of vertices, an **x,U-fan** is a set of paths from x to U such that any two of them share only the vertex x.

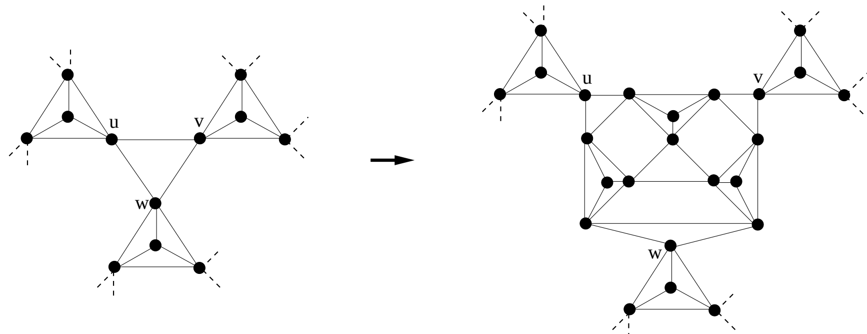

Figure 10: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 1.

Theorem 10 [3]: A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x and U with $|U| \geq k$, it has an x, U-fan of size k.

Technique 1: Let G be a graph in $\mathcal{G}$ on n K_4's containing a vertex of degree five, call it u. Then u must be a vertex of a K_4 that is adjacent to two other vertices on two other distinct K_4's, by Lemma 9. Call these neighbors of u in other K_4's v and w. By definition of $\mathcal{G}$, $v \sim w$. Delete the edges $\{uv, uw, vw\}$ from G and insert three new K_4's. Join each of u, v and w to two distinct exterior vertices on two distinct new K_4's so that each of u, v and w has edges to a unique pair of new K_4's, and add a set of edges necessary for claw-freedom. Each new K_4 then has one remaining exterior vertex of degree three; form a triangle with these vertices so that they all have degree five. Call the resulting graph G^+. (See Figure 10 for an illustration.) By Theorem 10, there exist three vertex disjoint paths from any vertex in $V(G) - \{u, v, w\}$ to the set $U = \{u, v, w\}$. Thus if x is a vertex of $V(G) - \{u, v, w\}$ and y is a new vertex, there exist three vertex disjoint paths from x to y, since the paths from x to U may be extended in a vertex disjoint way to y. It is left to the reader to check that there are three vertex disjoint paths in G^+ between any two vertices of U, between a vertex of U and a new vertex, and between any two new vertices. Thus G^+ is 3-connected. It is straightforward to check that G^+ is claw-free, planar and well-covered, and so G^+ is still a graph in $\mathcal{G}$, and it contains twelve more vertices than G does.

Technique 2: Let G be a graph in $\mathcal{G}$ on n K_4's containing a K_4 with two vertices of degree four; call them u and v. By Lemma 9, the fourth neighbors of u and v (i.e. the neighbors of u and v that are in another K_4) are distinct;

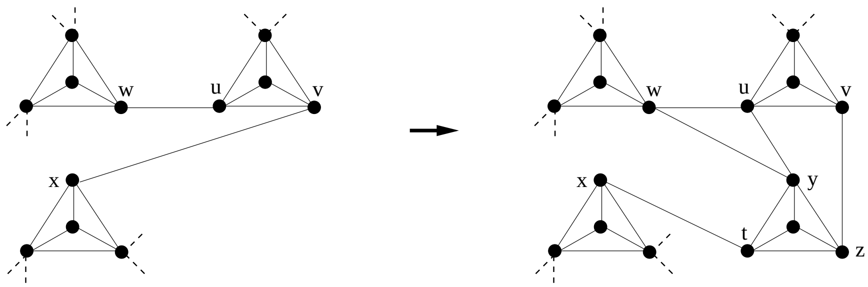

Figure 11: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 2.

call them w and x, such that $u \sim w$ and $v \sim x$. Note that w and x cannot be in the same K_4; otherwise the third exterior vertex of the K_4 containing u and v together with the third exterior vertex of the K_4 containing w and x would form a 2-cut, separating u, v, w, x and the two interior vertices of the two K_4's from the rest of the graph. Delete the edge vx from G and insert a new K_4. Let the exterior vertices of the new K_4 be y, z and t, and join the new K_4 with the following edges: uy, wy, zv, tx. Call the resulting graph G^+. (See Figure 11 for an illustration.) By Theorem 10, there exist three vertex disjoint paths from any vertex in $V(G) - \{u, v, x\}$ to the set $U = \{u, v, x\}$. Thus if s is a vertex of $V(G) - \{u, v, x\}$ and r is a new vertex, there exist three vertex disjoint paths from s to r, since the paths from s to U may be extended in a vertex disjoint way to r. It is left to the reader to check that there are three vertex disjoint paths in G^+ between any two vertices of U, between a vertex of U and a new vertex, and between any two new vertices. Thus G^+ is 3-connected. It is straightforward to check that G^+ is claw-free, planar and well-covered, and so G^+ is still a graph in $\mathcal{G}$, and it contains four more vertices than G does.

Technique 3: Let G be a graph in $\mathcal{G}$ on n K_4's containing at least three edges with the property that the vertices of the edge have degree four and the edge joins two K_4's (i.e. the edge is not contained within a K_4). Insert a new K_4, and join it to G so that each of the exterior neighbors of the new K_4 is joined to the two vertices of one of the edges of G with the property. (See Figure 12 for an illustration of building a graph in $\mathcal{G}$ with four K_4's from a graph of $\mathcal{G}$ with three K_4's using Technique 3.) The three edges we use to join the new K_4 are shown in bold in the original graph. Again we use Theorem 10 to show that the resulting graph is 3-connected. Here let U be a set of three vertices containing one vertex from each of the three edges with the property used to build the new graph. It is straightforward

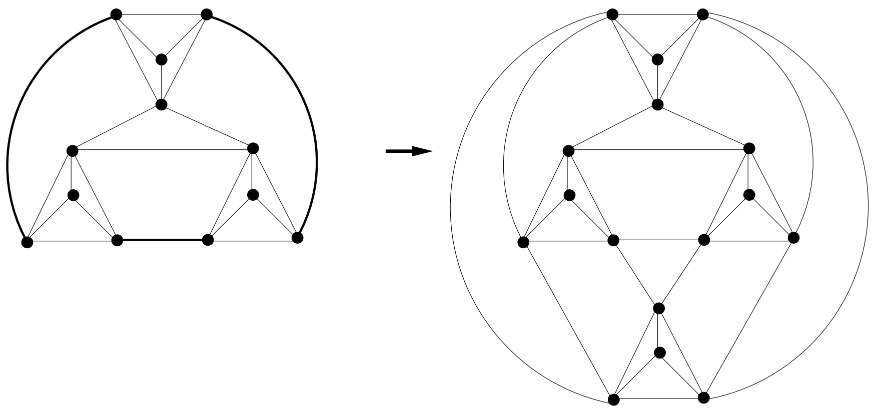

Figure 12: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 3.

to check that the new graph is claw-free, planar and well-covered, and so this graph is still a graph in $\mathcal{G}$, and it contains four more vertices than G does.

Note that by Lemma 8, each graph G in $\mathcal{G}$ either contains a vertex of degree five, or all the exterior vertices of the K_4's in G have degree four. Thus at least one of these three techniques can be applied to any given graph in $\mathcal{G}$ to build a larger graph (i.e. a graph with more vertices) in $\mathcal{G}$. Hence the class of graphs $\mathcal{G}$ contains an infinite number of graphs. Therefore there are an infinite number of planar, 3-connected, claw-free, well-covered (and well-dominated) graphs.

Now that we know which graphs are planar, 3-connected, claw-free and well-covered (or well-dominated), the obvious next question would be to relax one of these properties and see if we can still classify those graphs. An introduction to the author's research can be found in [5].

REFERENCES

1. S.R. Campbell and M.D. Plummer. On well-covered 3-polytopes. *Ars Combin.*, **25A** (1988) 215-242.

2. V. Chvátal and P.J. Slater. A note on well-covered graphs. *Quo Vadis, Graph Theory? Ann. Discrete Math.*, **55** (1993) 179-182.

3. G.A. Dirac. In abstrakten Graphen vorhande vollständigene 4-Graphen und ihre Unterteilungen. *Math. Nachr.*, **22** (1960) 61-85.

4. A. Finbow, B. Hartnell, and R.J. Nowakowski. Well-dominated graphs: a collection of well-covered ones. *Ars Combin.*, **25A** (1988) 5-10.

5. E.L.C. King. *Characterizing and Comparing Some Subclasses of Well-Covered Graphs.* Ph.D. Dissertation, Department of Mathematics, Vanderbilt University (2002).

6. M.D. Plummer. Claw-free maximal planar graphs. *Congr. Numer.*, **72** (1990) 9-23.

7. R.S. Sankaranarayana and L. K. Stewart. Complexity results for well-covered graphs. *Networks*, **22** (1992) 247-262.

8. D.B. West. *Introduction to Graph Theory, second edition.* Prentice-Hall, Inc. (2001).