NOTATION, TERMINOLOGY, AND PRELIMINARY RESULTS

For the most part, our notation and terminology is taken from West [20]. Let $G = (V, E)$ be a finite simple graph with vertex set V and edge set E, where each $e \in E$ is an unordered pair of distinct elements in V. If $e = uv$, then we say that u is *adjacent* to v or u and v are *neighbors*, and that e is *incident* to u and v. We will use the notation $u \sim v$ to denote that u is adjacent to v, and $u \nsim v$ to denote that u is not adjacent to v. A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of G is a subgraph with the added property that if $u, v \in V(H)$, then $uv \in E(H)$ if and only if $uv \in E(G)$. A neighbor w of v is called a *private* neighbor of v with respect to a set $S \subseteq V$ if no other vertex of S is adjacent to w. The *degree* of a vertex v, denoted $d(v)$, is equal to the number of edges that are incident with v. A graph is said to be *k-regular* if each vertex has degree k. The *open neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices of G that are adjacent to v. The *closed neighborhood* of a vertex v is $N[v] = N(v) \cup \{v\}$. An *independent set* $I \subseteq V$ of G is a set of vertices such that no two vertices of I are adjacent. The *independence number* of G, denoted $\alpha(G)$, is defined as the maximum cardinality of an independent set of G. We say that a graph is *well-covered* if every maximal independent set of G has the same cardinality. Determining whether or not a graph is well-covered has been shown to be co-NP-complete [3] [18].

A *path* is an alternating sequence of vertices and edges such that every edge joins the vertex preceding it with the vertex succeeding it and no vertex is repeated. If there is a path from every vertex in a graph G to every other vertex in G, then G is said to be *connected*; otherwise G is *disconnected*. A graph G is said to be *k-connected* if you must delete at least k vertices to disconnect G. If $G - v$ is a disconnected graph, then v is called a *cut-vertex.* A *bipartite* graph is one whose vertex set can be partitioned into two independent sets. A *complete bipartite* graph, denoted $K_{m,n}$ where one independent set A contains m vertices and the other independent set B contains n vertices, is a bipartite graph containing all possible edges joining a vertex from A to a vertex in B. Note that (A, B) is called a *bipartition* of the graph. A graph is said to be *claw-free* if it contains no induced $K_{1,3}$. A set of pairwise disjoint edges of a graph is called a *matching. Planar graphs* are those that can be drawn in the plane without any edges crossing.

A *dominating set* $D \subseteq V$ of G is a set such that each vertex $v \in V$ is either in the set or adjacent to a vertex in the set. The *domination number* of G, denoted $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. Note that $\gamma(G) \leq \alpha(G)$ for all graphs G, since any maximal independent set is a minimal dominating set. We say that a graph is *well-dominated* if every minimal dominating set of G has the same cardinality. Research in the area of well-dominated graphs was begun by Finbow, Hartnell and Nowakowski [5]. They were interested in characterizing different classes of well-covered graphs and were able to prove the following lemma.

Lemma 1.1 [5]: Every well-dominated graph is well-covered.

Hence they showed that the well-dominated graphs are a subclass of the well-covered graphs and hoped that perhaps they would be able to characterize this subclass. Unfortunately, this task has proved difficult. In Chapter II, we will characterize planar, claw-free, 3-connected, well-dominated graphs. Below are some lemmas that both describe some properties of well-dominated graphs and will assist us in characterizing these graphs.

Lemma 1.2: If G is a well-dominated graph and v is a vertex of G, then there exists a minimum dominating set containing v and a minimum dominating set not containing v.

Proof: To obtain a dominating set containing v, place v in the set D, delete $N[v]$ from G and continue in this greedy fashion until there are no vertices left. Then D is minimal and since G is well-dominated it is therefore minimum.

To obtain a minimum dominating set **not** containing v, we use the same greedy method except we use a neighbor of v as our initial vertex in D. ■

Lemma 1.3: Suppose G is a well-dominated graph with a cut-vertex v. Let H_1,..., H_k be the components of $G - v$. Then each H_i for $i = 1, ..., k$ is well-dominated.

Proof: By way of contradiction, suppose there exists an H_i of $G - v$ that is not well-dominated. Then there exist minimal dominating sets D_i and D_{k+1} of H_i such that $|D_i| < |D_{k+1}|$. Let u be a vertex of H_1 that is adjacent to v. Greedily choose a minimal dominating set, D_1, of H_1 containing u. Choose arbitrary minimal dominating sets D_j for each H_j where $j \neq 1, i$. Then $D = \bigcup_{j=1}^{k} D_j$ and $D^* = \left(\bigcup_{j=1}^{i-1} D_j\right) \cup \left(\bigcup_{j=i+1}^{k+1} D_j\right)$ are both minimal dominating sets of G, but $|D| < |D^*|$. This contradicts the fact that G is well-dominated. Hence each H_i is well-dominated. ■

It is important to note that Lemma 1.3 is **not** an if and only if statement. For

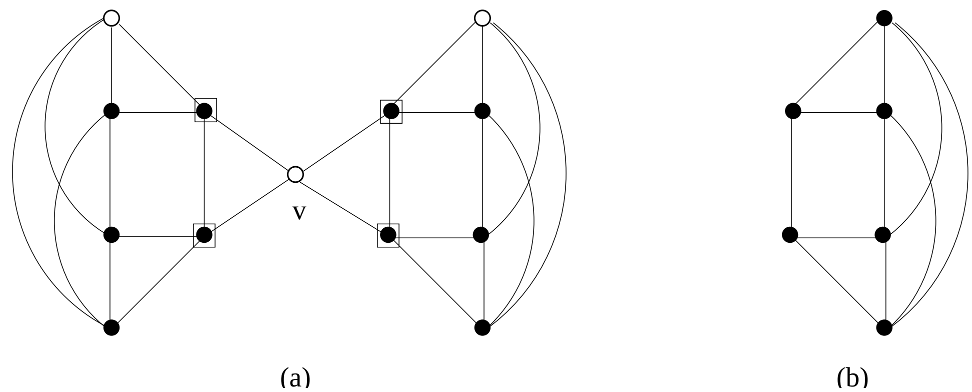

Figure 1: Although (b) is well-dominated, the graph shown in (a) is not well-dominated.

example, let G be the graph shown in (a) of Figure 1 with cut-vertex v. Then the component of $G - v$ shown in (b) of Figure 1 is well-dominated, but G is not. Two minimal dominating sets of G of different sizes are denoted by the white and boxed vertices respectively in Figure 1(a).

The following lemma, shown by Campbell and Plummer [2], is an important tool in proving the characterization theorems in Chapter II.

Lemma 1.4 [2]: Let G be a well-covered graph and I be an independent set of G. If C is a component of $G - N[I]$, then C is well-covered.

It is valuable to note that this technique for looking at well-covered graphs extends to well-dominated graphs as well.

Lemma 1.5: Let G be a graph and I be an independent set of G. Let C be a component of $G - N[I]$. If C is not well-dominated, then G is not well-dominated.

Proof: Let G be a graph, I be an independent set of G and C be a component of $G - N[I]$ that is not well dominated. Then there exist two minimal dominating sets of C, D_1 and D_2, such that $|D_1| < |D_2|$. Let D be a minimal dominating set of $G - N[I] - C$. Then $D \cup I \cup D_1$ and $D \cup I \cup D_2$ are minimal dominating sets of G and $|D \cup I \cup D_1| < |D \cup I \cup D_2|$. Therefore G is not well-dominated. ■

Often we have information that tells us how part of a graph, G, is defined, but not the entire graph. An induced subgraph of the graph on the vertices for which we have complete adjacency information at any given time in an argument we shall call a *partial*, P, of G. We define a vertex, x, of G to be a *link* vertex of P if x is contained in $G - P$ and is a neighbor in G of a vertex of P. Suppose u and v are two vertices of a partial P of G and we know both u and v have neighbors in G outside of P. Consider a link vertex y adjacent to u and a link vertex z adjacent to v. Since we do not have complete information about y and z it is possible that these two vertices could be identical and therefore that u and v share a neighbor outside of P. For this reason, we never assume that vertices we have labeled as link vertices are distinct. Let L be the set of link vertices associated with a partial P.

We can use these subgraphs of a graph, which we call partials, to determine whether or not the whole graph is well-dominated.

Additional terminology will be introduced when needed.

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