CHARACTERIZING PLANAR, 3-CONNECTED, CLAW-FREE, WELL-DOMINATED GRAPHS

Since it has been difficult to characterize well-dominated graphs in general, we narrow our perspective and look at well-dominated graphs that also have the properties of being planar, 3-connected and claw-free. By the nature of the relationship between well-covered and well-dominated graphs described in Lemma 1.1, we see that knowing the characterization of well-covered graphs with certain properties can lead us to the characterization of well-dominated graphs with those properties. In this chapter we will characterize all planar, 3-connected, claw-free, well-covered graphs and conclude that nearly all of them are well-dominated. To characterize these graphs, we will break our proof down into subcases determined by the possible degree of a given vertex. In that cause, we will find the following theorem helpful.

Theorem 2.1 [16]: If G is 3-connected, claw-free and planar, then

(a) $d(v) \leq 6$ for all $v \in V(G)$, and

(b) if v has degree 6 in G, then v lies on at least two separating triangles.

Given this theorem, we have only a finite number of possibilities for the degree of a vertex in a graph G that is 3-connected, claw-free, planar and well-covered. Considering all possibilities for the degree of a vertex v, we will explore the graph by proceeding from v and stopping regularly to check that all of the hypotheses still hold. Usually we will be considering only a portion of G (that portion to which we have "traveled" and can "see") and so we will use partials and link vertices as discussed in Chapter I. In addition to the terminology described in Chapter I, we will use the following terms defined immediately below and after Lemma 2.2.

Definition: A *semi-known* subgraph of G is a partial subgraph union some link vertices (those that are known) and edges and possibly edges between link vertices, though these are not necessarily present.

Definition: In the proofs to follow, at a given stage we say that a link vertex v of a semi-known subgraph S of a graph G can *grow* if it is adjacent to a link vertex of S in G via an edge not in S or if it is adjacent in G to a vertex of $V(G) - V(S)$. We say that S can *grow* if any of its vertices can grow.

Our proof will make extensive use of Lemma 1.4. We will regularly choose an independent set I of vertices in our semi-known subgraph in such a way that there is a component of $S - N[I]$ that is not well-covered. By Lemma 1.4, there cannot be such a component remaining when we delete an independent set from G, and thus there must be a vertex of S that can grow. The following lemma describes the different possibilities for the growth of S.

Lemma 2.2: Let G be a well-covered graph and S be a semi-known subgraph of G. Suppose that I is an independent set of vertices of S and C is a component of $S - N[I]$. If C is not well-covered, then either

(i) there is an edge, $e \notin E(S)$, in G between two vertices of I, i.e. I is dependent, or

(ii) there is an edge, $e \notin E(S)$, in G between two vertices of C,

or

(iii) there is an edge, $e \notin E(S)$, in G between a vertex of I and a vertex of C,

or

(iv) there is an edge, $e \notin E(S)$, in G between a vertex of C and a vertex of $V(G) - V(C) - I$ that is not adjacent to any vertex in I.

Proof: Assume the hypotheses of the lemma hold. Suppose by way of contradiction that C is not well-covered and none of (i), (ii), (iii), or (iv) hold. Since (i) does not hold, I is an independent set of G. Since (ii) and (iii) do not hold, C is a subgraph of $G - N[I]$. Since (iv) does not hold, C is not a proper subgraph of any larger connected subgraph of $G - N[I]$. Thus C is a component of $G - N[I]$. By Lemma 1.4, G is not well-covered, a contradiction. Therefore at least one of (i), (ii), (iii), or (iv) must hold. ∎

The following term is used when the fourth possibility of Lemma 2.2 occurs.

Definition: Let S be a semi-known subgraph of a graph G, I be a set of vertices in S that is independent in G, and C be a component of $S - N[I]$ that is not well-covered. Then any vertex v of $G - S$ that is adjacent to a vertex of C is said to be *born* by the deletion of $N[I]$. Thus we also say that v is not adjacent to any vertices of I *by birth*.

The following lemma will be very useful in eliminating possible subcases from our proof.

Lemma 2.3: Let G be a planar, claw-free, 3-connected graph, and S be a semi-known subgraph of G. Suppose there exists a set $I \subseteq V(S)$ that is independent in G, and a component, C, of $S - N[I]$ consisting of a vertex, v, and a subset of its neighbors such that: (i) v cannot grow, and (ii) there are at least two neighbors of v in C that cannot grow and are independent from one another. Then G is not well-covered.

Proof: Let S be a semi-known subgraph of G. Let I be a set of vertices such that $I \subseteq V(S)$ and I is independent in G, where $S - N[I]$ has a component C containing a vertex v that cannot grow and a subset of its neighbors. Let v_1 and v_2 be neighbors of v in C, which cannot grow and such that $v_1 \nsim v_2$. Since v cannot grow, all the adjacencies of v in G are known and so $I \cup \{v\}$ is independent in G. Extend $I \cup \{v\}$ to a maximal independent set of G; call it J. Let $T = J - I - \{v\}$. Note that since C is a component of $S - N[I]$, there are no edges between the vertex set $\{v, v_1, v_2\}$ and vertices of T. Thus $I \cup T \cup \{v_1, v_2\}$ is also independent in G. If this set is maximal independent in G, call it J'; otherwise extend it to a maximal independent set J' in G. Then $|J'| \geq |J| + 1$ and so G is not well-covered. ■

Let us define $\mathcal{G}$ to be the class of graphs containing K_4, and those graphs formed by a collection of K_4's drawn in the plane and connected by edges joining exterior vertices of the K_4's in such a way that G is 3-connected, plane and has the following property: if an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$.

In figures of the semi-known subgraphs we will signify that a vertex is a link vertex by showing dotted lines extending from the link vertex to the unknown parts of G.

Now that we have some basic tools to tackle this characterization theorem, let us begin.

Theorem 2.4: Let G be a planar, 3-connected graph. Then G is claw-free and well-covered if and only if G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$.

Proof of Theorem 2.4:

Claim 2.4/1: If G is planar, 3-connected, claw-free and well-covered, then G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$.

Proof of Claim 2.4/1: Suppose that G is planar, 3-connected, claw-free and well-covered. We leave it to the reader to check that the graphs in Figures 1 and 2 have these properties.

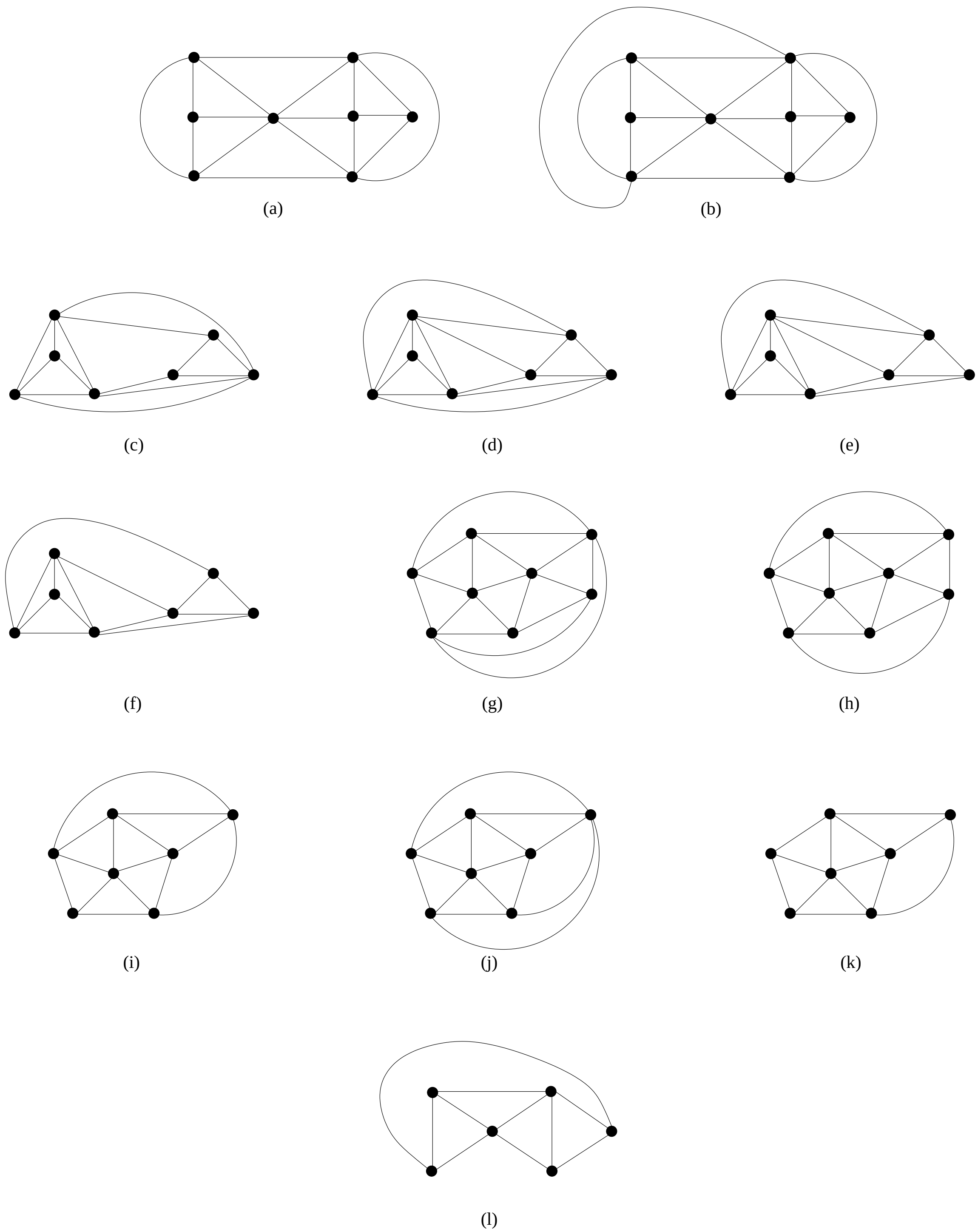


Figure 1: Exceptional well-covered, claw-free, planar, 3-connected graphs

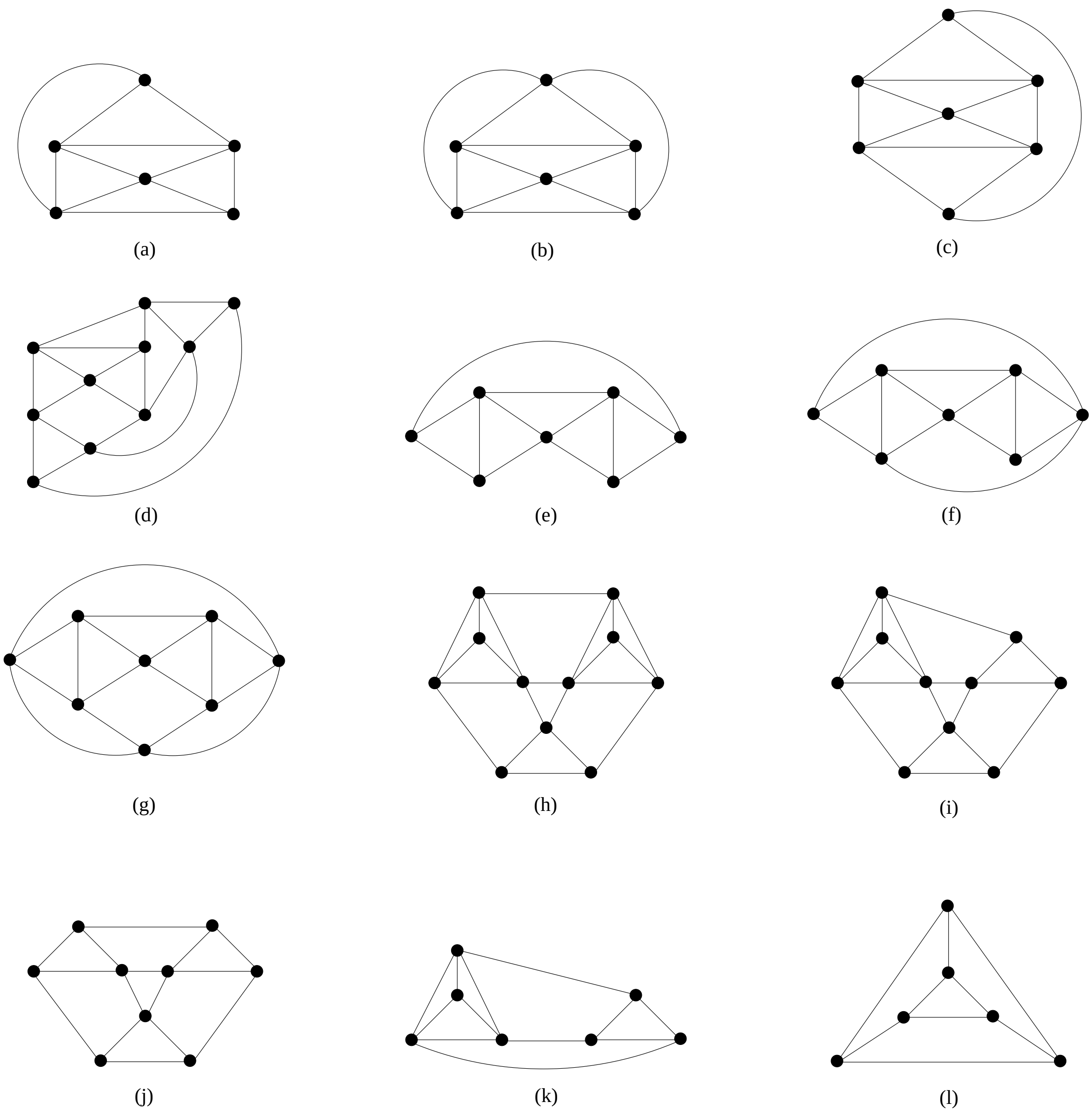


Figure 2: More exceptional well-covered, claw-free, planar, 3-connected graphs

Claim 2.4/1.1: If G has a vertex of degree six, then G is one of the first two exceptional graphs in Figure 1.

Proof of Claim 2.4/1.1: Let v be a vertex of G such that $d(v) = 6$. Label the neighbors of v in clockwise order: u, w, x, y, z, and t. By Theorem 2.1, v must lie on at least two separating triangles, and so without loss of generality, we may assume that $u \sim x$ and $y \sim t$. Then by claw-freedom, w is adjacent to u and x, and z is adjacent to y and t. Note that the induced subgraph S on $\{u, v, w, x, y, z, t\}$ is not well-covered, since $\{v\}$ and $\{z, w\}$ are both maximal independent sets of S. Therefore, S must not be all of G and at least one vertex of S must grow. By 3-connectivity, the four triangular faces having v as a corner vertex contain no additional vertices in G.

Claim 2.4/1.1.1: We must have at least two adjacencies between the vertex sets $\{u, x\}$, and $\{t, y\}$, and there are no vertices in the exterior face, that is, the $uxyt$-face.

Proof of Claim 2.4/1.1.1: Suppose that neither the uwx-face, nor the yzt-face contain vertices in G. Then $d(w) = 3 = d(z)$, and we can find an independent set I whose deletion leaves some subset of the vertices of $\{u, v, w, x, y, z, t\}$ containing at least $\{v, w, z\}$, the induced subgraph on which is not well-covered. By Lemma 1.4, this contradicts the fact that G is well-covered. Therefore, at least one of the uwx-face or the yzt-face contains an additional vertex. Suppose, without loss of generality, that the uwx-face contains at least one additional vertex. By 3-connectivity, each of u, w and x must then be adjacent to a vertex inside the uwx-face. Then u must not have any additional exterior neighbors, that is, neighbors in the $uxyt$-face; otherwise the additional exterior neighbor of u along with v and the additional interior neighbor of u forms a claw, contradicting the fact that G is claw-free. Similarly, x cannot have any additional exterior neighbors. But v is not a cut-vertex, and therefore each of u and x must be adjacent to at least one of y and t. Without loss of generality,

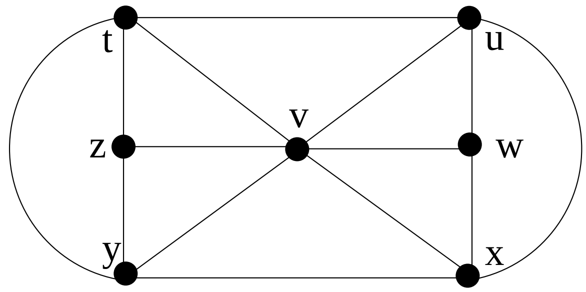


Figure 3: The local structure for a semi-known subgraph with $d(v) = 6$.

suppose that $u \sim t$ and $x \sim y$. By 3-connectivity, both the uvt-face and the xyv-face contain no additional vertices in G. By claw-freedom, t does not have any additional exterior neighbors; otherwise z, u and the additional exterior neighbor of t form a claw. Similarly, y cannot have any additional exterior neighbors. Therefore we may assume that we have $u \sim t$, $x \sim y$ and no vertices in the exterior face. ∎

By Claim 2.4/1.1.1, we may assume we have the semi-known subgraph S as shown in Figure 3. Note that this graph is not well-covered. The only additional edges that we may have between known vertices are uy or xt. By planarity, we cannot have both of these edges, so without loss of generality, suppose the only possible additional edge between known vertices is uy. Even if we add this edge, the resulting graph is not well-covered. Thus a vertex of S must grow into either the uwx-face or the yzt-face. Without loss of generality, suppose either u, w or x is adjacent to a new vertex in the uwx-face. By 3-connectivity, if one of u, w or x is adjacent to an additional vertex in the uwx-face then each of u, w and x is adjacent to an additional vertex in the uwx-face. Hence u is adjacent to an additional vertex in the uwx-face; call it s. To prevent $\{t, w, s\}$ and $\{t, x, s\}$ from forming claws with u, we must have that $w \sim s$ and $x \sim s$. This graph, with or without the additional edge uy is well-covered, and thus we have the first two exceptional graphs shown in Figure 1.

We will now show that G cannot contain any additional vertices and so must be one of these two graphs.

Claim 2.4/1.1.2: The graph G must contain exactly eight vertices.

Proof of Claim 2.4/1.1.2: Above it was shown that G cannot contain fewer than eight vertices. We now argue that no more vertices may be added to the first two graphs in Figure 1 to obtain larger well-covered graphs containing a vertex of degree six. Let S be a semi-known subgraph of G and one of the first two graphs in Figure 1. Suppose, by way of contradiction, we can add vertices to S to form a larger planar, 3-connected, claw-free, well-covered graph. By claw-freedom, there are no additional vertices in the usx-face; otherwise by 3-connectivity u would have an additional neighbor in this face and there would be a claw at u with t, w and the additional neighbor of u. Recall by Claim 2.4/1.1.1, there are no vertices in the exterior face. Thus, any additional vertices must be in either the yzt-face, the uws-face, or in the xws-face. Note that by 3-connectivity, if there exists an additional vertex in one of these faces all three vertices that make up the corners of this face must have a neighbor inside the face. Thus there cannot be additional vertices in both the uws-face and the xws-face, or v together with an additional neighbor of w from each of these faces forms a claw at w. If the edge uy is not in our graph, the uws-face is symmetric to the xws-face. If the edge uy is in our graph, then u already has six neighbors and so by Theorem 2.1 cannot have any neighbors in the uws-face. Thus without loss of generality, we may suppose that any additional vertex must be either in the yzt-face or in the xws-face, and no additional neighbors are in the uws-face.

Claim 2.4/1.1.2.1: There are no additional vertices in the xws-face.

Proof of Claim 2.4/1.1.2.1: Suppose, by way of contradiction, that there are n additional vertices in the xws-face. First we will show that there is only one way that x, w and s can grow into the xws-face to form adjacencies with these n additional vertices. The resulting subgraphs, known as *345-nests*, were described by Plummer

[16].

Claim 2.4/1.1.2.1.1: The n additional vertices form a 345-nest.

Proof of Claim 2.4/1.1.2.1.1: Let T_1 be the triangle formed by x, w and s. Note that $d(x) = 5$, $d(w) = 4$ and $d(s) = 3$. This is what Plummer [16] calls a *345-triangle*. By 3-connectivity, if there is one additional vertex in the xws-face, then each of x, w and s must be adjacent to a vertex in the xws-face. Let r_1 be the vertex in the xws-face that is adjacent to x. To prevent $\{y, w, r_1\}$ and $\{y, s, r_1\}$ from forming claws with x, we must have that $w \sim r_1$ and $s \sim r_1$. Note that x now has degree six, so by Theorem 2.1 and 3-connectivity, no additional vertices may appear in either the xsr_1-face or the xwr_1-face, and any additional vertices must be in the wsr_1-face. Suppose $n \geq 2$ and let r_2 be the vertex in the wsr_1-face that is adjacent to w. To prevent $\{v, s, r_2\}$ and $\{v, r_1, r_2\}$ from forming claws with w, we must have that $s \sim r_2$ and $r_1 \sim r_2$. Note that w now has degree six, so by Theorem 2.1 and 3-connectivity, no additional vertices may appear in either the wsr_2-face or the wr_1r_2-face, and any additional vertices must be in the sr_1r_2-face. Suppose $n \geq 3$ and let r_3 be the vertex in the sr_1r_2-face that is adjacent to s. To prevent $\{u, r_1, r_3\}$ and $\{u, r_2, r_3\}$ from forming claws with s, we must have that $r_1 \sim r_3$ and $r_2 \sim r_3$. The degree of r_1 is now five, and the degrees of r_2 and r_3 are four and three, respectively. Let T_2 be the triangle formed by r_1, r_2 and r_3. Note that this is also a 345-triangle. See Figure 4 for the graph with $n = 3$. And so we may continue adding a vertex r_i into a $r_{i-3}r_{i-2}r_{i-1}$-face, such that r_i is adjacent to all three of r_{i-3}, r_{i-2} and r_{i-1}. Every triangle formed by the vertices r_{3k-2}, r_{3k-1}, and r_{3k} where k is an integer such that $1 \leq k \leq \dfrac{n}{3}$ is a 345-triangle, T_k. Thus additional vertices create what Plummer [16] calls a *345-nest*, and this is the only way to grow x, s and w into the xws-face. ■

Claim 2.4/1.1.2.1.2: If G is well-covered, there cannot be a 345-nest in the xws-face.

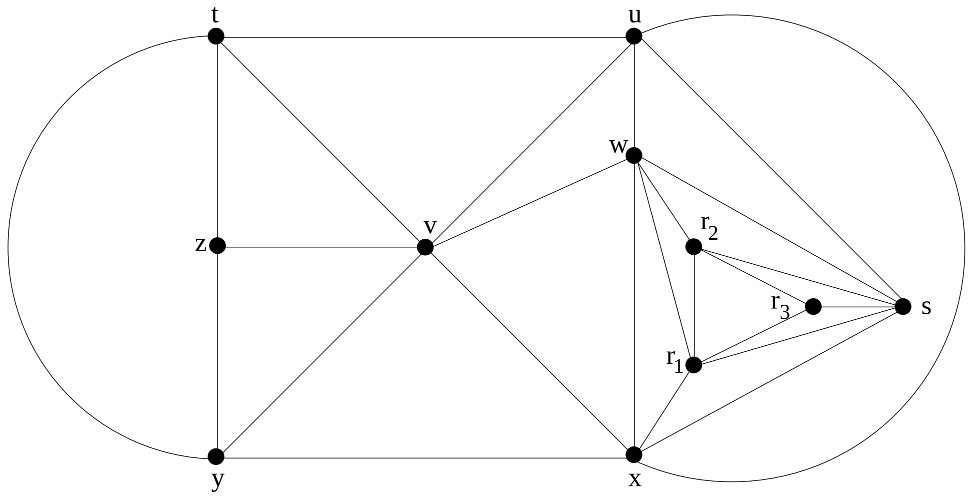


Figure 4: The semi-known subgraph S with a 345-nest when $n = 3$.

Proof of Claim 2.4/1.1.2.1.2: Suppose the n vertices in the xws-face of the semi-known graph S form a 345-nest with w, x and s as described in Claim 2.4/1.1.2.1.1. We will use Lemma 1.4. For $n = 4$ we will let the I in Lemma 1.4 be $I = \{v\}$ and for all other n, $n \geq 1$, $n \neq 4$, we will let $I = \{z\}$. If we can show that the component of $S - N[z]$ (or $S - N[v]$) containing the additional n vertices is not well-covered, this implies by Lemma 1.4 that G is not well-covered, which is a contradiction. For $1 \leq n \leq 6$, we will explicitly list in the table below the vertices in the component C of $S - N[I]$ as well as two maximal independent sets of C that are of different cardinalities. This shows that G is not well-covered for $n \leq 6$, a contradiction.

n	Independent Set	$V(C)$	Two Maximal I-sets
1	z	$\{u, w, x, s, r_1\}$	$\{s\}$, $\{u, r_1\}$
2	z	$\{u, w, x, s, r_1, r_2\}$	$\{s\}$, $\{u, r_1\}$
3	z	$\{u, w, x, s, r_1, r_2, r_3\}$	$\{s\}$, $\{u, r_1\}$
4	v	$\{s, r_1, r_2, r_3, r_4\}$	$\{r_3\}$, $\{s, r_4\}$
5	z	$\{u, w, x, s, r_1, r_2, r_3, r_4, r_5\}$	$\{u, r_4\}$, $\{u, r_1, r_5\}$
6	z	$\{u, w, x, s, r_1, r_2, r_3, r_4, r_5, r_6\}$	$\{x, r_3\}$, $\{x, r_2, r_6\}$

This proves that if S is a semi-known subgraph of G with a 345-nest as described above, then $n \geq 7$. Look at the component C of $S - N[z]$ containing the n additional vertices. The vertex set of C is $V(C) = \{u, w, x, s, r_1, ..., r_n\}$. Note that r_7 is adjacent to all the vertices of T_3 but none of the vertices of T_1 or T_2. Also note that r_7 is adjacent to $\{r_8, r_9, r_{10}\}$ if these vertices exist. Let I be a maximal independent set of the component of $S - N[r_7]$ containing r_{11}, if r_{11} exists, or let I be the empty set if r_{11} does not exist (i.e. if $7 \leq n \leq 10$). Then $I_1 = I \cup \{r_7\} \cup \{s\}$ and $I_2 = I \cup \{r_7\} \cup \{x\} \cup \{r_2\}$ are maximal independent sets of C of different cardinalities and hence G is not well-covered by Lemma 1.4, a contradiction. Therefore if G is well-covered, there cannot be a 345-nest in the xws-face. ■

By Claim 2.4/1.1.2.1.1, any additional vertices in the xws-face must form a 345-nest. However by Claim 2.4/1.1.2.1.2, any such 345-nest cannot be part of a well-covered graph with a vertex of degree six. Hence there are no additional vertices in the xws-face or in the uws-face, and we have proved Claim 2.4/1.1.2.1. ■

Claim 2.4/1.1.2.2: There are no additional vertices in the yzt-face.

Proof of Claim 2.4/1.1.2.2: Suppose, by way of contradiction, that there is at least one additional vertex in the yzt-face. By 3-connectivity, each of y, z and t is adjacent to an additional vertex in the yzt-face. Let r be the vertex in the yzt-face to which y is adjacent. To prevent $\{x, z, r\}$ and $\{x, t, r\}$ from forming claws with y, we must have that $z \sim r$ and $t \sim r$. Call this semi-known subgraph S. Then S is not well-covered since $\{z, w\}$ and $\{r, v, s\}$ are both maximal independent sets of S. Thus if there are graphs on more than eight vertices that have a vertex of degree six and are well-covered, S must grow. By claw-freedom, there are no additional vertices in the yrt-face; otherwise by 3-connectivity y would have an additional neighbor in the yrt-face and there would be a claw at y with x, z and the additional neighbor of y. By symmetry and Claim 2.4/1.1.2.1, there cannot be any additional vertices in either the yzr-face or the ztr-face. Therefore S cannot grow, a contradiction. ■

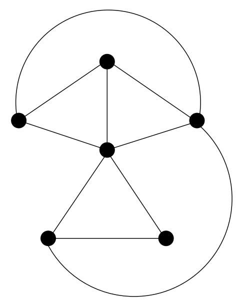
By Claims 2.4/1.1.2.1 and 2.4/1.1.2.2, we have shown that adding vertices to either of the first two graphs in Figure 1 will not yield a planar, 3-connected, claw-free, well-covered graph. Hence we have proved Claim 2.4/1.1.2, and the graph G, containing a vertex of degree six, must have exactly eight vertices. ■

Therefore we have proved Claim 2.4/1.1. If G has a vertex of degree six, then G is one of the first two exceptional graphs in Figure 1. ■

We may now assume that G is a planar, 3-connected, claw-free, well-covered graph, with $d(v) \leq 5$ for all $v \in V(G)$. The following claim will be useful.

Claim 2.4/1.2: Let G be a 3-connected, claw-free, plane graph, with the property that $d(v) \leq 5$ for all v in G. Suppose v is a vertex of G that is interior to a 3-cycle, where v is adjacent to all three of the vertices on the boundary of this cycle, and one of the boundary vertices has degree five (or degree four and cannot grow) such that two (or one) of its neighbors are not adjacent to v and the other boundary vertices. Then v cannot grow and has $d(v) = 3$.

Proof of Claim 2.4/1.2: Suppose G and v fulfill the hypotheses of the claim. Let u, w and x be the boundary vertices of the cycle, such that each of u, w and x is adjacent to v, and without loss of generality, suppose $d(u) = 5$ (or $d(u) = 4$ and u cannot grow). Then the two neighbors (or one neighbor) of u that are not adjacent to v or the other boundary vertices are exterior to the uwx-cycle by 3-connectivity. There can be no additional vertices in either the uvw-face or the uvx-face; otherwise $\{v, w\}$ or $\{v, x\}$ respectively would be 2-cuts, separating the additional vertices from the rest of the graph and contradicting the fact that G is 3-connected. Suppose there is an additional vertex in the vwx-cycle. Then by 3-connectivity, each of v, w and x must have a neighbor interior to this cycle. Thus w cannot have an additional neighbor exterior to the uwx-cycle; otherwise this additional exterior neighbor together with u and the additional neighbor in the vwx-cycle would be a claw at w, contradicting the fact that G is claw-free. But then $\{u, x\}$ is a 2-cut, separating v and w from the



(a) (b)

Figure 5: Two forbidden subgraphs containing a vertex of degree five.

rest of the graph and contradicting the fact that G is 3-connected. Hence there is no vertex in the vwx-face either. Therefore v cannot grow and has $d(v) = 3$. ■

Claim 2.4/1.3: If G is not one of the exceptional graphs in Figure 1 or Figure 2, then every vertex of G must lie on a K_4.

Proof of Claim 2.4/1.3: Suppose that G is not one of the exceptional graphs in Figure 1 or Figure 2. Let v be a vertex of G. By Theorem 2.1 and Claim 2.4/1.1, $d(v) < 6$. Since G is 3-connected, $d(v) \geq 3$. Thus we must show that if $3 \leq d(v) \leq 5$, then v lies on a K_4.

First we will show that, unless G is isomorphic to one of the three graphs in Figure 7, the two subgraphs in Figure 5 are forbidden, even though the vertex of degree five in each is on a K_4. Note that the graphs in Figure 7 all belong to the class $\mathcal{G}$.

Suppose v is an arbitrary vertex of G with degree 5 that lies on a K_4. Label the neighbors of v in a clockwise fashion: u, z, w, y, x. Without loss of generality, suppose that u, z, and w form a K_4 with v, such that z is the interior vertex of the K_4. To prevent $\{vz, vx, vy\}$ from forming a claw, we must have $x \sim y$.

Claim 2.4/1.3.1: The graph shown in Figure 5(a) is a forbidden subgraph in the

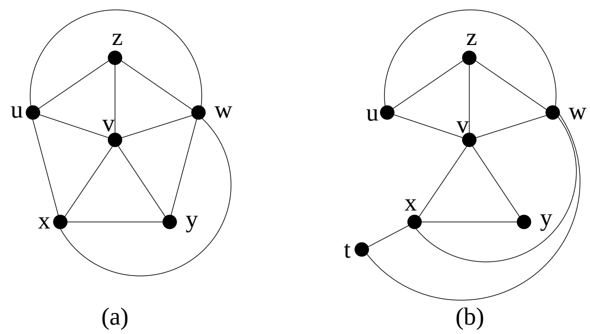


Figure 6: Proving the subgraph in Figure 5(a) is forbidden.

graph G.

Proof of Claim 2.4/1.3.1: By way of contradiction, suppose that $w \sim x$, giving us the subgraph in Figure 5(a) centered at v. Since G is 3-connected, $\{v, w\}$ is not a 2-cut and so there must be a path from u to x. Either u is adjacent to x or x is adjacent to an additional vertex in the exterior face.

Suppose that $u \sim x$. Since G is 3-connected, $\{v, x\}$ is not a 2-cut, and so there must be a path from y to w. Either w is adjacent to y, or w is adjacent to an additional vertex inside the $wvyx$-face.

Suppose $w \sim y$. Then $d(w) = 5$ and by 3-connectivity, the graph, as shown in Figure 6(a), cannot grow. But this graph is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets. Hence, this graph is not G and so we may assume that $w \nsim y$.

Suppose that w is adjacent to an additional vertex; call it t, inside the $wvyx$-face. To prevent $\{wz, wx, wt\}$ from forming a claw, we must have $x \sim t$. Now $d(x) = 5 = d(w)$ and so x and w cannot have any additional neighbors. If any additional vertices were in the $vwtxy$-face then $\{y, t\}$ would be a 2-cut, contradicting the fact that G is 3-connected. Thus since $\{w, x\}$ is not a 2-cut, there must be a path from t to y, and since the path cannot use any additional vertices, we have

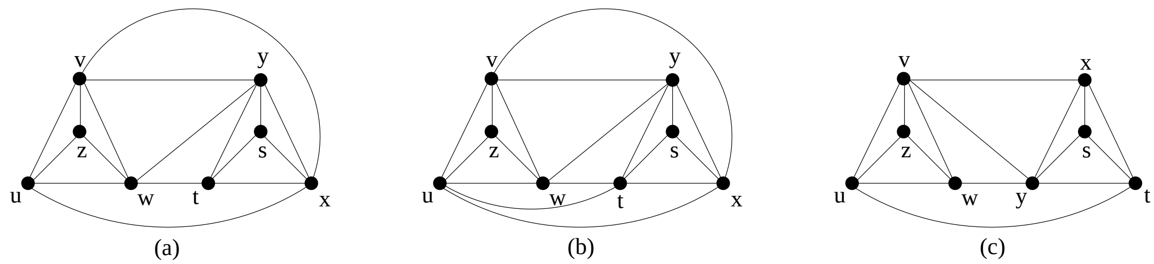


Figure 7: The only members of $\mathcal{G}$ containing the subgraph shown in Figure 5(b).

$t \sim y$. By 3-connectivity, this graph cannot grow. This graph is well-covered and is shown in Figure 1(c). Since G is not a graph in from this figure, we may assume w is not adjacent to an additional vertex and hence $u \nsim x$.

Suppose that x is adjacent to an additional vertex in the exterior face; call it t. To prevent $\{xt, xy, xw\}$ from forming a claw, either $w \sim y$ or $w \sim t$. Suppose that $w \sim y$. Then $d(w) = 5$, and so w can have no additional neighbors. But then $\{u, x\}$ is a 2-cut, separating t from the rest of the graph and contradicting the fact that G is 3-connected. Thus $w \nsim y$.

Suppose that $w \sim t$. Then $d(w) = 5$ and so w can have no additional neighbors in the graph shown in Figure 6(b). But then $\{v, x\}$ is a 2-cut, separating y from the rest of the graph and contradicting the fact that G is 3-connected. Thus $w \nsim t$ and so x is not adjacent to an additional vertex.

Therefore $w \nsim x$ and similarly $u \nsim y$. Thus the graph shown in Figure 5(a) is a forbidden subgraph in G. ■

Claim 2.4/1.3.2: The graph shown in Figure 5(b) is a forbidden subgraph in the graph G, unless G is isomorphic to one of the graphs in Figure 7.

Proof of Claim 2.4/1.3.2: By way of contradiction, suppose that $w \sim y$, giving us the subgraph in Figure 5(b) centered at v. Either w is adjacent to an additional vertex in the exterior face, w is adjacent to a vertex in the uzw-face, or $d(w) = 4$.

Claim 2.4/1.3.2.1: The vertex w cannot be adjacent to an additional vertex in the exterior face.

Proof of Claim 2.4/1.3.2.1: Suppose, by way of contradiction, that w is adjacent to an additional vertex in the exterior face; call it t. To prevent $\{wz, wy, wt\}$ from forming a claw, we must have $y \sim t$. Since G is 3-connected, $\{v, w\}$ is not a 2-cut and there must be a path from u to the vertex set $\{x, y, t\}$. Either u is adjacent to x, u is adjacent to t, or u is adjacent to an additional vertex. Recall that by Claim 2.4/1.3.1, $u \nsim y$, since this would give the forbidden subgraph in Figure 5(a).

Claim 2.4/1.3.2.1.1: The vertex u cannot be adjacent to x.

Proof of Claim 2.4/1.3.2.1.1: Suppose, by way of contradiction, that $u \sim x$. Since $u \nsim y$, either y is adjacent to an additional vertex or $d(y) = 4$.

Suppose that y is adjacent to an additional vertex; call it s. To prevent $\{yv, yt, ys\}$ from forming a claw, we must have $t \sim s$. To prevent $\{yw, yx, ys\}$ from forming a claw, we must have $x \sim s$. Either x is adjacent to t, x is adjacent to an additional vertex, or $d(x) = 4$. Suppose that $x \sim t$. Then we have the graph as shown in Figure 7(a). Note that every vertex in this graph lies on a K_4, and that this is a graph in the class $\mathcal{G}$. By 3-connectivity, either this is all of G or $u \sim t$. If $u \sim t$, then we have the graph as shown in Figure 7(b). Every vertex in this graph lies on a K_4 and this is a graph in the class $\mathcal{G}$. By 3-connectivity, this graph cannot grow. So we may assume that $x \nsim t$. Suppose that x is adjacent to an additional vertex; call it r. To prevent $\{xv, xs, xr\}$ from forming a claw, we must have $s \sim r$. To prevent $\{xy, xu, xr\}$ from forming a claw, we must have $u \sim r$. This semi-known subgraph S is not well-covered since $\{s, z\}$ and $\{r, y, z\}$ are both maximal independent sets in S. Thus S must grow into the exterior face. By 3-connectivity, each of t, s and r must be adjacent to additional vertices in this face. Let q be an additional neighbor of s. By claw-freedom, we must have $q \sim t$ and $r \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v. Then C is not well-covered since $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C. Thus be Lemma 1.4, S is not a semi-known subgraph of G and thus x cannot be adjacent to an additional vertex. So we may assume that $d(x) = 4$. Let S be this semi-known subgraph. Let C be the component of $S - N[t]$ containing v. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and x are vertices of C that cannot grow, and $z \nsim x$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $d(y) = 4$.

Since G is 3-connected, $\{y, w\}$ is not a 2-cut, and so there must be a path from t to either u or x. Either u is adjacent to t, x is adjacent to t, or both u and x (by claw-freedom) share an additional neighbor. Suppose that $u \sim t$. To prevent $\{uz, ut, ux\}$ from forming a claw, we must have $x \sim t$. Then the graph is well-covered and is the exceptional graph shown in Figure 1(d). By 3-connectivity, this graph cannot grow. Since G is not one of the graphs in Figure 1, we may assume that $u \nsim t$. Suppose that $x \sim t$. Then the graph is well-covered and is the exceptional graph shown in Figure 1(e). Since G is not one of the graphs in Figure 1, this graph must grow. Then each of u, x and t must be adjacent to an additional vertex. Let s be an additional neighbor of x. To prevent $\{xs, xv, xt\}$ from forming a claw, we must have $s \sim t$. To prevent $\{xy, xu, xs\}$ from forming a claw, we must have $s \sim u$. Then $d(u) = 5 = d(x)$ and so by 3-connectivity, the graph can grow no more. But it is not well-covered since $\{s, w\}$ and $\{s, z, y\}$ are both maximal independent sets. Hence $x \nsim t$. Suppose that both u and x share an additional neighbor, call it s. Now $d(u) = 5$, y cannot grow, and $u \nsim y$ so x cannot grow by claw-freedom (and the fact that $x \nsim t$ by the preceding case). Therefore there can be no additional vertices in the exterior face or $\{s, t\}$ would be a 2-cut, separating these additional vertices from the rest of the graph and contradicting the fact that G is 3-connected. But $\{w, y\}$ cannot be a 2-cut (separating t from the rest of the graph) either, so $s \sim t$. This graph cannot grow, but it is not well-covered since $\{s, v\}$ and $\{s, z, y\}$ are both maximal independent sets of the graph. Therefore $u \nsim x$. ■

Claim 2.4/1.3.2.1.2: The vertex u cannot be adjacent to t.

Proof of Claim 2.4/1.3.2.1.2: Suppose, by way of contradiction, that $u \sim t$. Recall that $u \nsim y$, since then the graph would contain the forbidden subgraph shown in Figure 5(a). Also $u \nsim x$ by Claim 2.4/1.3.2.1.1. Hence either u is adjacent to an additional vertex, or $d(u) = 4$.

Suppose that u is adjacent to an additional vertex; call it s. To prevent $\{uz, ut, us\}$ from forming a claw, we must have $s \sim t$. Either y is adjacent to s, y is adjacent to an additional vertex, or $d(y) = 4$. Suppose $y \sim s$. To prevent $\{yw, yx, ys\}$ from forming a claw, we must have $x \sim s$. Then by 3-connectivity, the graph cannot grow. But it is not well-covered since $\{s, v\}$ and $\{x, z, t\}$. Hence we may assume that $y \nsim s$. Suppose that y is adjacent to an additional vertex; call it r. To prevent $\{yw, yx, yr\}$ from forming a claw, we must have $x \sim r$. To prevent $\{yv, yt, yr\}$ from forming a claw, we must have $t \sim r$. To prevent $\{tw, ts, tr\}$ from forming a claw, we must have $s \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and $d(y) = 4$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence u is not adjacent to an additional vertex.

Suppose $d(u) = 4$. Since G is 3-connected, $\{v, y\}$ is not a 2-cut, and so there must be a path from x to t that does not pass through either v or y. Either $x \sim t$ or x is adjacent to an additional vertex. Suppose $x \sim t$. Then the graph is well-covered and is the exceptional graph shown in Figure 1(e). Since G is not one of the graphs in Figure 1, the graph must grow. The only option is for additional vertices to be in the xyt-face. By 3-connectivity, each of x, y and t must be adjacent to a vertex in this face. Let s be an additional neighbor of y. To prevent $\{yv, yt, ys\}$ from forming a claw, we must have $t \sim s$. To prevent $\{yw, yx, ys\}$ from forming a claw, we must have $x \sim s$. Then we have the graph as shown in Figure 7(a). Note that every vertex in this graph lies on a K_4, and that this is a graph in the class $\mathcal{G}$. By 3-connectivity, this graph cannot grow, and so we may assume that $x \nsim t$. Suppose that x is adjacent to an additional vertex. Then by 3-connectivity, y must be adjacent to an additional vertex; call it s. To prevent $\{yx, yw, ys\}$ from forming a claw, we must have $x \sim s$. To prevent $\{yv, yt, ys\}$ from forming a claw, we must have $t \sim s$. Then,since $x \nsim t$, the degree of t must be four; otherwise any additional neighbor of t would form a claw with u and y, since $u \nsim y$ by Claim 2.4/1.3.1. Then the graph cannot grow by 3-connectivity. But it is not well-covered since $\{s, z\}$ and $\{x, t, z\}$ are both maximal independent sets. Therefore $u \nsim t$. ■

Claim 2.4/1.3.2.1.3: The vertex u cannot be adjacent to an additional vertex.

Proof of Claim 2.4/1.3.2.1.3: Suppose, by way of contradiction, that u is adjacent to an additional vertex s. Either y is adjacent to s, y is adjacent to an additional vertex, or $d(y) = 4$.

Suppose $y \sim s$. To prevent $\{yv, yt, ys\}$ from forming a claw, we must have $s \sim t$. To prevent $\{yw, yx, ys\}$ from forming a claw, we must have $s \sim x$. This graph, call it S, is not well-covered since $\{z, y\}$ and $\{x, z, t\}$ are both maximal independent sets, so it must grow. There are no possible additional edges between known vertices, so there must be additional vertices. Now $d(y) = 5$ so either there is an additional

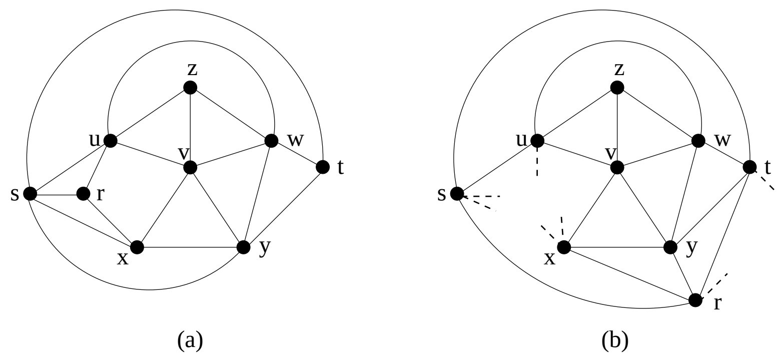


Figure 8: Proving the subgraph in Figure 5(b) is forbidden.

vertex in the $uvxs$-face, or there is an additional vertex in the $uwts$-face. Suppose there is an additional vertex in the $uvxs$-face. Since G is 3-connected, each of u, x, and s must have a neighbor in this face. Let r be the neighbor of u. To prevent $\{uz, ur, us\}$ from forming a claw, we must have $s \sim r$. To prevent $\{st, sr, sx\}$ from forming a claw, we must have $x \sim r$. Now $d(s) = 5 = d(u)$ and the graph, as shown in Figure 8(a), cannot grow. But it is not well-covered since $\{s, w\}$ and $\{r, z, t\}$ are both maximal independent sets. Thus we may assume there is no additional vertex in the $uvxs$-face. Suppose there is an additional vertex in the $uwts$-face. Since G is 3-connected, each of u, s, and t must have a neighbor in this face. Let r be the neighbor of u. To prevent $\{uz, ur, us\}$ from forming a claw, we must have $s \sim r$. To prevent $\{st, sr, sx\}$ from forming a claw, we must have $t \sim r$. Now $d(s) = 5 = d(u)$ and the graph cannot grow. But it is not well-covered since $\{r, v\}$ and $\{r, z, y\}$ are both maximal independent sets. Therefore S cannot grow, which is a contradiction since G is well-covered. Hence $y \nsim s$.

Suppose y is adjacent to an additional vertex; call it r. To prevent $\{yv, yt, yr\}$ from forming a claw, we must have $t \sim r$. To prevent $\{yw, yx, yr\}$ from forming a claw, we must have $x \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[s, t]$ containing v. Then $V(C) = \{z, v, x\}$. By Lemma 2.2, either $s \sim t$, $s \sim x$, or x is adjacent to an additional vertex. Suppose $s \sim t$. To prevent $\{tw, ts, tr\}$ from forming a claw, we must have $s \sim r$. Now call this semi-known subgraph S, as shown in Figure 8(b). Let C be the component of $S - N[s]$ containing v. Then $V(C) = \{v, w, y, z, t\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $s \nsim t$. Suppose $s \sim x$. To prevent $\{xs, xv, xr\}$ from forming a claw, we must have $s \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing w. Then $V(C) = \{v, w, y, z, t\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $s \nsim x$. Suppose that x is adjacent to an additional vertex; call it q. To prevent $\{xq, xv, xr\}$ from forming a claw, we must have $q \sim r$. Note that $s \nsim q$ by birth. Now call this semi-known subgraph S. Let C be the component of $S - N[s, q]$ containing w. Then $V(C) = \{v, w, y, z, t\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore y is not adjacent to an additional vertex.

Suppose that $d(y) = 4$. Since G is 3-connected, $\{w, y\}$ is not a 2-cut, and so there must be a path from t to the vertex set $\{u, s, x\}$ that does not go through either w or y. Thus either t is adjacent to x, t is adjacent to s, or t is adjacent to a new vertex.

Suppose $t \sim x$. See the graph in Figure 9(a) for an illustration. Either t is adjacent to s, t is adjacent to a new vertex, or $d(t) = 3$. Suppose $t \sim s$. To prevent $\{tx, tw, ts\}$ from forming a claw, we must have $x \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v. Then $V(C) = \{v, w, y, z\}$, C cannot grow, and C is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets of C. Then G is not well-covered by Lemma 1.4, a contradiction.

Hence $t \nsim s$. Suppose t is adjacent to an additional vertex; call it r. Now call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing v. Then $V(C) = \{v, w, y, z, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus if $s \nsim r$, then by Lemma 2.3, G is not well-covered, a contradiction. So suppose $s \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v. Then $V(C) = \{u, v, w, y, z, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex and we may assume $d(t) = 3$. Since G is 3-connected, $\{v, w\}$ is not a 2-cut, so there must be a path from x to u or s that does not pass through v or w. However, $x \nsim u$ by Claim 2.4/1.3.2.1.1, and so any additional neighbor of x will form a claw with v and t, since v and t are independent and cannot grow. Therefore $t \nsim x$.

Suppose $t \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v. Then $V(C) = \{v, w, y, z, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $s \nsim t$.

Suppose that t is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing v. Then $V(C) = \{v, w, y, z, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus if $s \nsim r$, then by Lemma 2.3, G is not well-covered, a contradiction. So suppose $s \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v. Then $V(C) = \{u, v, w, y, z, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus $d(y) \neq 4$, and so u cannot be adjacent to an

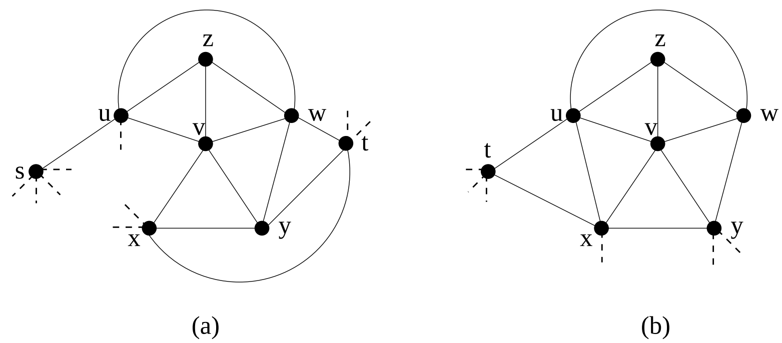


Figure 9: Proving the subgraph in Figure 5(b) is forbidden.

additional vertex. ■

Claim 2.4/1.3.2.1.1, Claim 2.4/1.3.2.1.2, and Claim 2.4/1.3.2.1.3 combine to prove that $d(u) = 2$, which contradicts the fact that G is 3-connected. Hence, w cannot be adjacent to an additional vertex in the exterior face, and we have proved Claim 2.4/1.3.2.1. ■

Claim 2.4/1.3.2.2: The vertex w cannot be adjacent to an additional vertex in the uzw-face.

Proof of Claim 2.4/1.3.2.2: Suppose, by way of contradiction, that G is adjacent to a vertex in the uzw-face; call it t. To prevent $\{wz, wt, wy\}$ from forming a claw, we must have $t \sim z$. To prevent $\{wu, wt, wy\}$ from forming a claw, we must have $t \sim u$. Since G is 3-connected, $\{v, w\}$ is not a 2-cut, and so there must be a path from u to either x or y that does not pass through the vertices v and w. Recall that $u \nsim y$; otherwise we would have the forbidden subgraph in Figure 5(a). Hence either u is adjacent to x or u is adjacent to an additional vertex in the exterior face. Suppose $u \sim x$. By 3-connectivity, this graph cannot grow. Then the graph is the exceptional graph in Figure 1(c). Since G is not one of the graphs in Figure 1, we may assume that $u \nsim x$. Then u must be adjacent to an additional exterior vertex. But then this new vertex along with v and t form a claw at u, contradicting the fact that G is claw-free. Hence w cannot be adjacent to an additional vertex in the uzw-face. ■

Claim 2.4/1.3.2.3: The degree of w must be greater than four.

Proof of Claim 2.4/1.3.2.3: Suppose, by way of contradiction, that $d(w) = 4$. Then by 3-connectivity, $d(z) = 3$. Since G is 3-connected, $\{v, w\}$ is not a 2-cut, and so there must be a path from u to x and y that does not pass through either v or w. Recall that $u \nsim y$; otherwise we would have the forbidden subgraph in Figure 5(a). Hence either u is adjacent to x or u is adjacent to an additional vertex.

Suppose that $u \sim x$. The resulting graph is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets, and so the graph must grow. No additional edges may be added between known vertices, so there must be at least one additional vertex. Then by 3-connectivity, each of x, y, and u must be adjacent to an additional vertex. Let t be the additional neighbor of u. To prevent $\{uz, ux, ut\}$ from forming a claw, we must have $x \sim t$. See the graph in Figure 9(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t]$. Then $V(C) = \{z, w, v, y\}$. By Lemma 2.2, either $y \sim t$ or y is adjacent to an additional vertex.

Suppose $y \sim t$. Then we have the exceptional well-covered graph shown in Figure 1(e). Since G is not one of the graphs in Figure 1, the graph must grow. There are no possible edges to add between known vertices in the graph, so there must be additional vertices in the txy-face. By 3-connectivity, each of t, x and y must be adjacent to an additional vertex in this face. Let s be the additional neighbor of x. To prevent $\{xy, xs, xu\}$ from forming a claw, we must have $s \sim y$. To prevent $\{xt, xs, xv\}$ from forming a claw, we must have $s \sim t$. Now the graph is a well-covered graph isomorphic to the graph in Figure 7(a). Since the graph can grow no further, we may assume that $y \nsim t$.

Suppose y is adjacent to an additional vertex; call it s. Since $d(w) = 4$, to prevent $\{yw, yx, ys\}$ from forming a claw at y, we must have $x \sim s$. Note that now $d(x) = 5$. To prevent $\{xv, xt, xs\}$ from forming a claw at x, we must have $t \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing v. Then $V(C) = \{v, w, y, z\}$. By Lemma 2.2, y must be adjacent to an additional vertex. But then this additional vertex along with x and w form a claw at y, contradicting the fact that G is claw-free. Hence y cannot grow so C cannot grow, and C is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets of C. Therefore by Lemma 1.4, G is not well-covered, a contradiction. Hence $u \nsim x$.

Suppose u is adjacent to an additional vertex; call it t. Since G is 3-connected, $\{u, x\}$ is not a 2-cut and so there must be a path from t to y that does not pass through these two vertices. Thus either y is adjacent to t or y is adjacent to an additional vertex.

Suppose $y \sim t$. Since $d(w) = 4$, to prevent $\{yw, yx, yt\}$ from forming a claw, we must have $x \sim t$. Then we have the exceptional well-covered graph shown in Figure 1(f). Since G is not one of the graphs in Figure 1, the graph must grow. There are no possible edges to add between known vertices in the graph, so there must be at least one additional vertex in either the $uvxt$-face, the xyt-face, or the $wytu$-face. Suppose there is an additional vertex in the $uvxt$-face. Then by 3-connectivity, each of u, x, and t must be adjacent to a vertex in this face. Let s be an additional neighbor of u in the $uvxt$-face. To prevent $\{uz, ut, us\}$ from forming a claw at u, we must have $t \sim s$. Since G is 3-connected, either $x \sim s$ or x is adjacent to an additional vertex in the $uvxt$-face. Suppose $x \sim s$. By claw-freedom and 3-connectivity, this graph cannot grow. But it is not well-covered since $\{x, z\}$ and $\{s, y, z\}$ are both maximal independent sets. Hence we may assume that $x \nsim s$. Suppose that x is adjacent to a different additional vertex in the $uvxt$-face; call it r. To prevent $\{xr, xv, xt\}$ from forming a claw at x, we must have $r \sim t$. But then since $d(u) = 5$, we have a claw at t with $\{tu, ty, tr\}$, a contradiction. Hence we may assume that there is no additional vertex in the $uvxt$-face. Suppose there is an additional vertex in the xyt-face. Then by 3-connectivity, each of x, y, and t must be adjacent to a vertex in this face. Let s be an additional neighbor of y in the xyt-face. To prevent $\{yw, yx, ys\}$ from forming a claw at y, we must have $x \sim s$. To prevent $\{yv, yt, ys\}$ from forming a claw at y, we must have $t \sim s$. Now the graph is a well-covered graph isomorphic to the graph in Figure 7(c). By claw-freedom and 3-connectivity, this graph cannot grow, and so we may assume that there is no additional vertex in the xyt-face. Suppose there is an additional vertex in the $wytu$-face. Then by 3-connectivity, each of y, t and u must be adjacent to a vertex in this face. Let s be the additional neighbor of y in the $wytu$-face. But then $\{yw, yx, ys\}$ is a claw at y, contradicting the fact that G is claw-free. Hence $y \nsim t$.

Suppose that y is adjacent to an additional vertex; call it s. To prevent $\{yw, yx, ys\}$ from forming a claw at y, we must have $x \sim s$. Either y is adjacent to an additional vertex in the $uwysxv$-face, y is adjacent to an additional vertex in the xys-face, or $d(y) = 4$. Suppose y is adjacent to an additional vertex r in the exterior face. To prevent $\{yw, yr, ys\}$ from forming a claw at y, we must have $r \sim s$. To prevent $\{yw, yx, yr\}$ from forming a claw at y, we must have $x \sim r$. Since G is 3-connected, $\{u, r\}$ is not a 2-cut, and so there must be a path from x to t that does not pass through either u or r. Hence either $x \sim t$ or x is adjacent to an additional vertex in the $uwyrxv$-face. In either case, t or the additional vertex to which x is adjacent along with v and s form a claw at x, contradicting the fact that G is claw-free. Thus y is not adjacent to an additional vertex in the $uwysxv$-face. Suppose y is adjacent to an additional vertex in the xys-face; call it r. To prevent $\{yw, yr, ys\}$ from forming a claw at y, we must have $r \sim s$. To prevent $\{yw, yx, yr\}$ from forming a claw at y, we must have $x \sim r$. Since G is 3-connected, $\{u, s\}$ is not a 2-cut, and so there must be a path from x to t that does not pass through either u or s. Hence either $x \sim t$ or x is adjacent to an additional vertex in the $uwysxv$-face. In either case, t or the additional vertex to which x is adjacent along with v and r form a claw at x, contradicting the fact that G is claw-free. Hence y is not adjacent to an additional vertex in the xys-face, and we may assume $d(y) = 4$. Since G is 3-connected, $\{u, s\}$ is not a 2-cut, and so there must be a path from x to t that does not pass through either u or s. Hence either $x \sim t$ or x is adjacent to an additional vertex. Suppose that $x \sim t$. To prevent $\{xv, xs, xt\}$ from forming a claw at x, we must have $s \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing v, so that $V(C) = \{z, w, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $x \nsim t$. Suppose x is adjacent to an additional vertex; call it r. To prevent $\{xv, xs, xr\}$ from forming a claw at x, we must have $s \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r]$ containing v, so that $V(C) = \{z, w, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{z, y\}$ are both maximal independent sets of C. Thus, if $t \nsim r$, by Lemma 1.4, G is not well-covered, a contradiction. So suppose $t \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, z, w, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so u is not adjacent to an additional vertex, and therefore the degree of w must be greater than 4. ∎

Suppose G is not isomorphic to one of the graphs in Figure 7. Then since Claim 2.4/1.3.2.3 shows that w must be adjacent to an additional vertex and Claim 2.4/1.3.2.1 together with Claim 2.4/1.3.2.2 shows that w cannot be adjacent to an additional vertex, we have a contradiction. Hence the graph shown in Figure 5(b) is a forbidden subgraph in the graph G, and we have proved Claim 2.4/1.3.2. ∎

Using the fact that the subgraphs in Figure 5 are forbidden, as proved in Claim 2.4/1.3.1 and Claim 2.4/1.3.2, we will now show that every vertex of G must lie on a K_4. We will break the cases down into the possible degrees a vertex may have.

Claim 2.4/1.3.3: If G is not one of the exceptional graphs in Figure 1 and v is a vertex of G with $d(v) = 5$, then v must lie on a K_4.

Proof of Claim 2.4/1.3.3: Suppose, by way of contradiction, that the hypotheses of the Claim are fulfilled but that v does not lie on a K_4. Since v does not lie on a K_4, without loss of generality, we may assume that v is the center of a 5-wheel. Label the neighbors of v in a clockwise fashion: u, z, w, y, x. This graph is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets. Hence the graph must grow. Since v does not lie on a K_4, there can be no additional edges between known vertices. Hence one of the vertices must be adjacent to an additional vertex. Since all the neighbors of v are isomorphic to each other, we may assume without loss of generality that w is adjacent to a new vertex, call it t. To prevent $\{wz, wt, wy\}$ from forming a claw, either $t \sim z$ or $t \sim y$. Without loss of generality, suppose that $t \sim z$. Either w is adjacent to an additional vertex or $d(w) = 4$.

Claim 2.4/1.3.3.1: The degree of w must be four.

Proof of Claim 2.4/1.3.3.1: Suppose, by way of contradiction, that w is adjacent to an additional vertex; call it s. If s is in the zwt-face, then by claw-freedom at w, s must be adjacent to both z and t resulting in the forbidden subgraph shown in Figure 5(b) and centered at w. Hence s must be in the exterior face. To prevent $\{wv, wt, ws\}$ from forming a claw at w, we must have $t \sim s$. To prevent $\{wz, wy, ws\}$ from forming a claw at w, we must have $y \sim s$, since $z \sim s$ would result in the forbidden subgraph shown in Figure 5(b) and centered at w. Then the graph appears as shown in Figure 10(a). Either y is adjacent to an additional vertex or $d(y) = 4$.

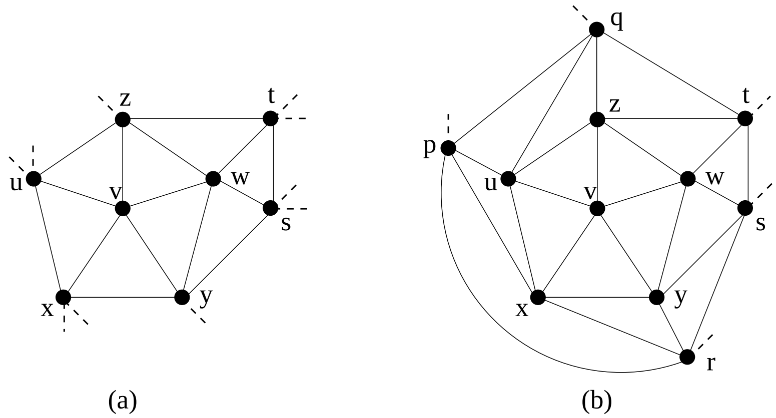


Figure 10: Proving that every vertex of degree five must lie on a K_4.

Suppose that y is adjacent to an additional vertex; call it r. To prevent $\{yv, yr, ys\}$ from forming a claw at y, we must have $r \sim s$. To prevent $\{yw, yx, yr\}$ from forming a claw at y, we must have $r \sim x$. Either z is adjacent to r, z is adjacent to an additional vertex, or $d(z) = 4$.

Suppose $z \sim r$. To prevent $\{zv, zt, zr\}$ from forming a claw at z, we must have $t \sim r$. But then $\{zr, zw, zu\}$ is a claw at z since $d(w) = 5 = d(r)$, a contradiction. Hence $z \nsim r$.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zv, zt, zq\}$ from forming a claw at z, we must have $t \sim q$. To prevent $\{zu, zw, zq\}$ from forming a claw at z, we must have $u \sim q$. Note that $u \nsim t$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at z. Hence either u is adjacent to s, u is adjacent to r, u is adjacent to an additional vertex, or $d(u) = 4$. Suppose that $u \sim s$. Then $\{ux, uz, us\}$ is a claw at u, since $x \nsim s$ otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at y. This contradicts the fact that G is claw-free and hence $u \nsim s$. Suppose that $u \sim r$. To prevent $\{uq, uv, ur\}$ from forming a claw at u, we must have $r \sim q$. To prevent $\{rx, rs, rq\}$ from forming a claw at r, we must have $s \sim q$. Since t and x are the only vertices with degree less than five and they are in different faces, by 3-connectivity this graph can grow no further. But it is not well-covered since $\{v, q\}$ and $\{x, w, q\}$ are both maximal independent sets. Hence $u \nsim r$. Suppose that u is adjacent to an additional vertex; call it p. To prevent $\{uv, uq, up\}$ from forming a claw at u, we must have $p \sim q$. To prevent $\{uz, ux, up\}$ from forming a claw at u, we must have $p \sim x$. To prevent $\{xp, xv, xr\}$ from forming a claw at u, we must have $p \sim r$. See the graph in Figure 10(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p]$ containing w, so that $V(C) = \{v, z, w, y, t, s\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence u is not adjacent to an additional vertex, and thus $d(u) = 4$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing z, so that $V(C) = \{v, z, w, u, t, q\}$. Then all the vertices of $C - z$ are adjacent to z, vertices z, u and w are vertices of C that cannot grow, and $w \nsim u$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex.

Suppose $d(z) = 4$. By symmetry, we also have $d(x) = 4$ and $d(s) = 4$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{v, z, w, u\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, x and z are vertices of C that cannot grow, and $x \nsim z$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore y is not adjacent to an additional vertex.

Suppose that $d(y) = 4$. By symmetry, we also have $d(z) = 4$. Either s is adjacent to x, s is adjacent to u, s is adjacent to an additional vertex, or $d(s) = 3$.

Suppose $s \sim x$. Either x is adjacent to t, x is adjacent to an additional vertex, or $d(x) = 4$. Suppose that $x \sim t$. To prevent $\{xu, xy, xt\}$ from forming a claw, we must have $u \sim t$. Then we have the exceptional well-covered graph shown in Figure

1(g). Since this graph cannot grow and G is not one of the graphs in Figure 1, we may assume that $x \nsim t$. Suppose that x is adjacent to an additional vertex; call it r. To prevent $\{xu, xr, xy\}$ from forming a claw at x, we must have $u \sim r$. To prevent $\{xv, xr, xs\}$ from forming a claw at x, we must have $s \sim r$. To prevent $\{st, sr, sy\}$ from forming a claw at s, we must have $t \sim r$. Call this semi-known subgraph S and see Figure 11(a) for an illustration. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{v, z, w, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex. Suppose $d(x) = 4$. By symmetry, we can also say $d(s) = 4$. Since G is 3-connected, we cannot add any additional vertices to this graph otherwise $\{u, t\}$ would be a 2-cut. However this graph is not well-covered since $\{y, z\}$ and $\{y, u, t\}$ are both maximal independent sets. Thus we must have $u \sim t$. Then we have the exceptional well-covered graph shown in Figure 1(h). Since this graph cannot grow and G is not one of the graphs in Figure 1, we may assume that $s \nsim x$.

Suppose $s \sim u$. To prevent $\{st, su, sy\}$ from forming a claw at s, we must have $u \sim t$. But then $\{uz, us, ux\}$ is a claw at u, since z cannot grow and $x \nsim s$. This contradicts the fact that G is claw-free and thus $s \nsim u$.

Suppose that s is adjacent to an additional vertex; call it r. To prevent $\{sy, st, sr\}$ from forming a claw at s, we must have $r \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, z, w, y, x\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, z and y are vertices of C that cannot grow, and $z \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex.

Suppose that $d(s) = 3$. Then by symmetry $d(x) = d(u) = d(t) = 3$ and thus the graph cannot grow. But it is not well-covered since $\{v, s\}$ and $\{u, t, y\}$ are both

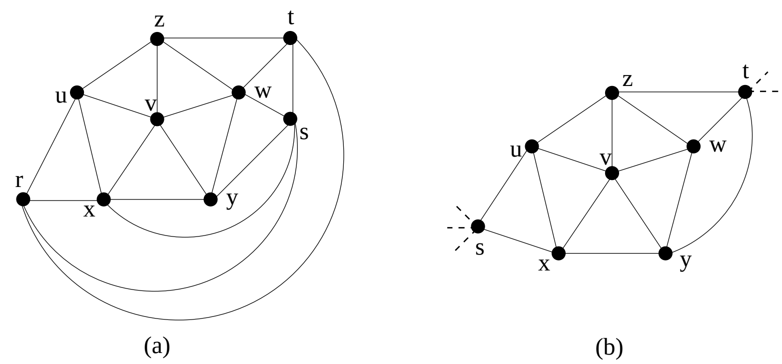


Figure 11: Proving that every vertex of degree five must lie on a K_4.

maximal independent sets of G. Therefore w is not adjacent to an additional vertex so $d(w) = 4$, and any vertex adjacent to v has degree either three or four. ∎

Claim 2.4/1.3.3.2: The degree of w must be greater than four.

Proof of Claim 2.4/1.3.3.2: Suppose, by way of contradiction, that $d(w) = 4$. By Claim 2.4/1.3.3.1, each of u, z, w, y and x has degree either three or four. Thus $d(z) = 4$. Either y is adjacent to t, y is adjacent to an additional vertex, or $d(y) = 3$.

Suppose $y \sim t$, then $d(y) = 4$. Then this graph is the exceptional well-covered graph shown in Figure 1(k). Since G is not one of the graphs in Figure 1, this graph must grow. Either u is adjacent to t, u is adjacent to an additional vertex, or $d(u) = 3$. Suppose $u \sim t$, then $d(u) = 4$. Then this graph is the exceptional well-covered graph as shown in Figure 1(i). Since G is not one of the graphs in Figure 1, this graph must grow. Since G is 3-connected, we cannot add any additional vertices; otherwise $\{x, t\}$ would be a 2-cut. Hence the only way the graph can grow is to add an edge from x to t. So suppose $x \sim t$. Then this graph is the exceptional well-covered graph shown in Figure 1(j). Since this graph cannot grow and G is not one of the graphs in Figure 1, we may assume that $u \nsim t$. Suppose u is adjacent to an additional vertex; call it s. To prevent $\{uz, ux, us\}$ from forming a claw at u, we must have $s \sim x$. Then $d(u) = d(x) = 4$, and we have the graph as shown in Figure 11(b). Since G is 3-connected, we cannot add any additional vertices; otherwise $\{s, t\}$ would be a 2-cut. However $\{u, x\}$ is now a 2-cut, so there must be a path from s to t that does not pass through u or x. Hence $s \sim t$. But then $\{tz, ts, ty\}$ is a claw at t, contradicting the fact that G is claw-free. Thus u is not adjacent to an additional vertex, and $d(u) = 3$. By symmetry, we also know that $d(x) = 3$. Then again this graph is the exceptional well-covered graph shown in Figure 1 (k). Since G is 3-connected, we cannot add any vertices to this graph; otherwise t would be a cut-vertex. Since this graph cannot grow and G is not one of the graphs in Figure 1, we may assume that $y \nsim t$.

Suppose y is adjacent to an additional vertex; call it s. To prevent $\{yw, yx, ys\}$ from forming a claw at y, we must have $x \sim s$. Then $d(y) = d(x) = 4$. If u is adjacent to either s or t, then we have a subgraph isomorphic to the one in the u adjacent to an additional vertex subcase of the $y \sim t$ case above (see Figure 11(b)). Hence we may assume that u is not adjacent to either s or t. Then either u is adjacent to an additional vertex or $d(u) = 3$. Suppose that u is adjacent to an additional vertex; call it r. But then $\{ur, uz, ux\}$ is a claw since neither x nor z may grow. Hence u is not adjacent to an additional vertex and $d(u) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing z, so that $V(C) = \{u, v, z, w, t\}$. Then all the vertices of $C - z$ are adjacent to z, vertices z, u and w are vertices of C that cannot grow, and $u \nsim w$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex.

Suppose $d(y) = 3$. By symmetry we also know $d(u) = 3$. Since G is 3-connected, we cannot add any additional vertices to this graph or $\{x, t\}$ would be a 2-cut. But $\{z, w\}$ is now a 2-cut, and so there must be a path from t to x not passing through either z or w. Hence $x \sim t$, and this graph cannot grow. But it is not well-covered since $\{t, v\}$ and $\{t, u, y\}$ are both maximal independent sets. Hence the degree of w must be greater than four. $\blacksquare$

Claim 2.4/1.3.3.1 says that $d(w) = 4$ and Claim 2.4/1.3.3.2 says that $d(w)$ must be greater than four. This is a contradiction. Therefore, if G is not one of the exceptional graphs in Figure 1 and v is a vertex of G with $d(v) = 5$, then v must lie on a K_4. $\blacksquare$

Note that in proving Claim 2.4/1.3.3, we also showed that, unless G is one of the exceptional graphs in Figure 1, the 5-wheel is a forbidden subgraph in G.

Claim 2.4/1.3.4: If G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 4$, then v must lie on a K_4.

Proof of Claim 2.4/1.3.4: Let G be a graph and v be a vertex of G that fulfill the hypotheses of the claim. Label the neighbors of v in a clockwise fashion: u, x, y, w. Suppose, by way of contradiction, that v does not lie on a K_4. By claw-freedom, and without loss of generality, we may assume that $x \sim y$ and $u \sim w$. We call this subgraph, the *bow-tie subgraph* centered at v. Since G is 3-connected, every vertex of G has degree at least three. Thus x must grow. Either x is adjacent to u, x is adjacent to w or x is adjacent to an additional vertex.

Claim 2.4/1.3.4.1: The vertex x is not adjacent to u.

Proof of Claim 2.4/1.3.4.1: Suppose, by way of contradiction, that $x \sim u$. Then $x \nsim w$ and $u \nsim y$; otherwise v would lie on a K_4. Thus either x is adjacent to an additional vertex or $d(x) = 3$. Suppose that x is adjacent to an additional vertex; call it z. To prevent $\{xu, xy, xz\}$ from forming a claw at x, either $z \sim u$ or $z \sim y$.

Claim 2.4/1.3.4.1.1: The vertex z is not adjacent to u.

Proof of Claim 2.4/1.3.4.1.1: Suppose, by way of contradiction, that $z \sim u$.

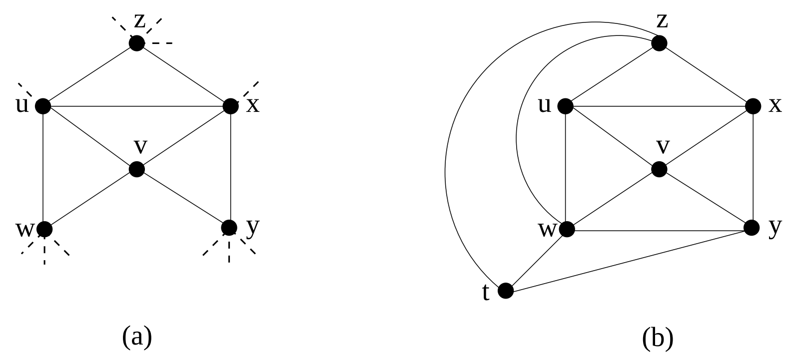


Figure 12: Proving that every vertex of degree four must lie on a K_4.

See Figure 12(a) for an illustration. Either x is adjacent to an additional vertex or $d(x) = 4$. Suppose x is adjacent to an additional vertex; call it t. Suppose t is in the uxz-face. Then by claw-freedom, $t \sim u$ and $t \sim z$, and we have the forbidden subgraph shown in Figure 5(b) and centered at x or u. Hence t must be in the exterior face. To prevent $\{xv, xz, xt\}$ from forming a claw at x, we must have $z \sim t$. Now x is a vertex of degree five and so by Claim 2.4/1.3.3, x must lie on a K_4. Since the 5-wheel is a forbidden subgraph, $t \nsim y$. Thus, since v does not lie on a K_4, x must be in a K_4 with u, z and t, and so $u \sim t$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at x or u. Hence x is not adjacent to an additional vertex.

Suppose $d(x) = 4$. Then by symmetry, we have $d(u) = 4$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must be a path from w to either y or z that does not pass through u or v. Thus either w is adjacent to y, w is adjacent to z, or w is adjacent to an additional vertex.

Claim 2.4/1.3.4.1.1.1: The vertex w is not adjacent to y.

Proof of Claim 2.4/1.3.4.1.1.1: Suppose, by way of contradiction, that $w \sim y$. Since G is connected, $\{u, x\}$ is not a 2-cut and so there must be a path from z to w or y that does not pass through u or x. Then either z is adjacent to w, z is adjacent to

y, or w and y are both adjacent to an additional vertex. (Note that if w is adjacent to an additional vertex then y must be adjacent to that same additional vertex; otherwise the additional neighbor of w, along with y and u would create a claw at w. A similar argument shows that if y is adjacent to an additional vertex then w must be adjacent to the same additional vertex.) Suppose $z \sim w$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(a). Since G is not one of the graphs in Figure 2, the graph must grow. Either w is adjacent to an additional vertex or $d(w) = 4$. Suppose that w is adjacent to an additional vertex; call it t. To prevent $\{wv, wz, wt\}$ from forming a claw at w, we must have $z \sim t$. To prevent $\{wu, wy, wt\}$ from forming a claw at w, we must have $y \sim t$. See the graph in Figure 12(b) for an illustration. Then we have a forbidden 5-wheel centered at w. Therefore, w is not adjacent to an additional vertex. Suppose $d(w) = 4$. Recall that this graph is the exceptional well-covered graph shown in Figure 2(a), and thus the graph must grow. Since G is 3-connected, we cannot add any additional vertices or $\{z, y\}$ would be a 2-cut. Hence our only option for growth is to add the edge zy. Then the resulting graph is isomorphic to the exceptional well-covered graph shown in Figure 2(b). Since G is not one of the graphs in Figure 2, $z \nsim w$.

Suppose $z \sim y$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(a). Since G is not one of the graphs in Figure 2, this graph must grow. Since $z \nsim w$ by the preceding case, we cannot add any additional edges to the graph. Thus we must add a vertex, which must be in the $wyzu$-face by 3-connectivity. Also by 3-connectivity, each of w, y and z must be adjacent to a vertex in this face. Let t be an additional neighbor of y in the $wyzu$-face. To prevent $\{yx, yw, yt\}$ from forming a claw at y, we must have $w \sim t$. To prevent $\{yt, yv, yz\}$ from forming a claw at y, we must have $z \sim t$. Then we have a forbidden 5-wheel centered at y. Therefore, $z \nsim y$.

Suppose w and y are share an additional neighbor, call it t. Either w is adjacent to an additional vertex or $d(w) = 4$. Suppose w is adjacent to an additional vertex; call it s. To prevent $\{wu, wy, ws\}$and $\{wu, wt, ws\}$ from forming claws at w, we must have $y \sim s$ and $t \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w or y. Hence w is not adjacent to an additional vertex. Suppose $d(w) = 4$. Then, by symmetry, $d(y) = 4$. Since G is 3-connected, we cannot add any more vertices to this graph or $\{z, t\}$ would be a 2-cut. But now $\{w, y\}$ is a 2-cut, and so we must have a path from z to t that does not pass through w or y. Thus $z \sim t$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(c). Since G is not one of the graphs in Figure 2, w and y cannot share an additional neighbor, and so $w \nsim y$. ■

Claim 2.4/1.3.4.1.1.2: The vertex w is not adjacent to z.

Proof of Claim 2.4/1.3.4.1.1.2: Suppose, by way of contradiction, that $w \sim z$. Since G is 3-connected, $\{v, x\}$ is not a 2-cut and so there must be a path from y to either z or w that does not pass through v or x. Thus, since $w \nsim y$, either y is adjacent to z, or z and w share an additional vertex. (Note that if w is adjacent to an additional vertex then z must be adjacent to that same additional vertex; otherwise the additional neighbor of w, along with z and v would create a claw at w. A similar argument shows that if z is adjacent to an additional vertex then w must be adjacent to the same additional vertex.) Suppose that $y \sim z$. Then again this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(a). Since G is not one of the graphs in Figure 2, this graph must grow. However the only way to grow the graph is to add an additional vertex. By claw-freedom, this vertex would be adjacent to w, z and y, but then this would create a forbidden 5-wheel centered at z. Hence $y \nsim z$.

Suppose that z and w share an additional neighbor, call it t. Since G is 3-

connected, $\{z, w\}$ is not a 2-cut and so there must be a path from t to y that does not pass through either z or w. Hence either t is adjacent to y or t is adjacent to an additional vertex in the $zxyvw$-face. Suppose $t \sim y$. Then again this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(c). Since G is not one of the graphs in Figure 2, the graph must grow. The only option for growth is for additional vertices to be added. By 3-connectivity, at least one of z or w must be adjacent to at least one additional vertex; call it s. But if one of z or w is adjacent to s, then the other must be also to prevent either $\{zs, zx, zw\}$ or $\{ws, wz, wv\}$ from forming a claw. Hence z and w must share an additional neighbor, s, in the zwt-face. To prevent $\{wv, wt, ws\}$ from forming a claw at w, we must also have $t \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w or z. Hence w and z are not adjacent to additional vertices and so $t \nsim y$. Suppose t is adjacent to an additional vertex in the $zxyvw$-face; call it s. By 3-connectivity, at least one of z or w must be adjacent to at least one additional vertex inside the $zxyvw$-face. But if one is adjacent to at least one additional vertex inside the $zxyvw$-face, then the other must also be so adjacent to prevent either $\{zr, zx, zw\}$ or $\{wr, wz, wv\}$ from forming a claw where r is the additional vertex to which they are adjacent. Thus either both z and w are adjacent to s, or z and w are both adjacent to an additional vertex. Suppose both z and w are adjacent to s. But then we have the forbidden subgraph shown in Figure 5(a) and centered at z. Hence z and w are not adjacent to s and z and w must share an additional neighbor, call it r. To prevent $\{zt, zr, zu\}$ from forming a claw at z, we must have $r \sim t$. But then $d(z) = 5 = d(w)$ and so $\{r, t\}$ is a 2-cut separating s from the rest of the graph. This contradicts the fact that G is 3-connected and hence $w \nsim z$. ■

Note that by symmetry and since $w \nsim z$, we can also now assume that $y \nsim z$.

Claim 2.4/1.3.4.1.1.3: The vertex w is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.1.3: Suppose, by way of contradiction, that w is adjacent to an additional vertex; call it t. Since G is 3-connected, $\{v, x\}$ is not a 2-cut and so there must be a path from y to z or w that does not pass through either v or x. Thus since $y \nsim w$ and $y \nsim z$, either y is adjacent to t, or y is adjacent to an additional vertex.

Claim 2.4/1.3.4.1.1.3.1: The vertex y is not adjacent to t.

Proof of Claim 2.4/1.3.4.1.1.3.1: Suppose, by way of contradiction, that $y \sim t$. Since G is 3-connected, $\{w, y\}$ is not a 2-cut and so there must be a path from z to t that does not pass through these two vertices. Thus either z is adjacent to t or z is adjacent to an additional vertex. Suppose $z \sim t$. But then $\{tw, ty, tz\}$ is a claw at t, contradicting the fact that G is claw-free. Hence $z \nsim t$. Suppose z is adjacent to an additional vertex; call it s. Since G is 3-connected, $\{t, z\}$ is not a 2-cut, and so there must be a path from s to either w or y that does not pass through either t or z. Thus either y (or symmetrically w) is adjacent to s, or y (or symmetrically w) is adjacent to an additional vertex.

Suppose $y \sim s$. To prevent $\{yv, yt, ys\}$ from forming a claw at y, we must have $s \sim t$. Call this semi-known subgraph S. See Figure 13(a) for an illustration. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since both $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C, and so by Lemma 2.2, either $s \sim w$ or w is adjacent to an additional vertex.

Suppose $s \sim w$. Then $\{sw, sy, sz\}$ is a claw, contradicting the fact that G is claw-free. Hence $s \nsim w$.

Suppose w is adjacent to an additional vertex; call it r. To prevent $\{wu, wt, wr\}$ from forming a claw at w, we must have $t \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing v, so that $V(C) = \{u, v, x, z\}$.

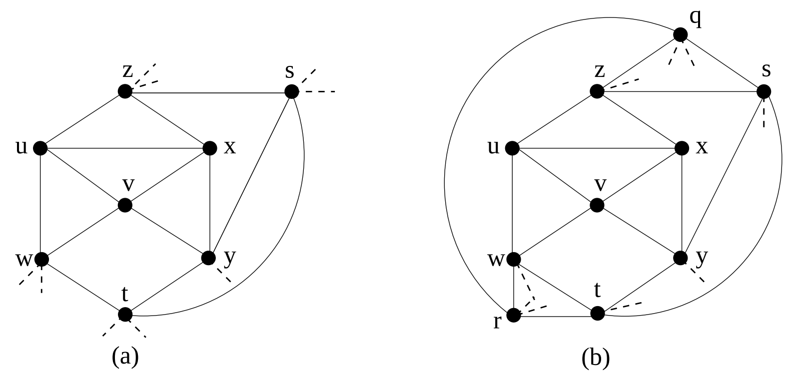


Figure 13: Proving that every vertex of degree four must lie on a K_4.

Then C is not well-covered since both $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C, and so by Lemma 2.2 and since $t \nsim z$ (by a previous subcase among the subcases of Claim 2.4/1.3.4.1.1.3.1), z must be adjacent to an additional vertex; call it q. To prevent $\{zq, zu, zs\}$ from forming a claw at z, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since both $\{v\}$ and $\{u, y\}$ are maximal independent sets of C, and so by Lemma 2.2, either $q \sim r$ or y is adjacent to an additional vertex. (Note that by planarity, $y \nsim q$ and $y \nsim r$.)

Suppose $q \sim r$. See Figure 13(b) for an illustration. Then this graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. Note that z cannot have any additional neighbors in the $zqrwu$-face or the additional neighbor along with x and s would form a claw at z, contradicting the fact that G is claw-free. Also w cannot have any additional neighbors in the $zqrwu$-face or the additional neighbor along with v and t would form a claw at w, contradicting the fact that G is claw-free. Hence since G is 3-connected and u cannot grow, there can be no additional vertices in the $zqrwu$-face. Note that t cannot have any additional neighbors in the $qrts$-face or the additional neighbor along with w and y would form a claw at t, contradicting the fact that

G is claw-free. Also s cannot have any additional neighbors in the $qrts$-face or the additional neighbor along with z and y would form a claw at s, contradicting the fact that G is claw-free. Hence since G is 3-connected, there can be no additional vertices in the $qrts$-face. Thus we may have additional vertices only in the wrt-face, the yst-face, or the zqs-face. Note that $t \nsim q$ by birth. Given our previous results and these observations we have the following possibilities for the graph to grow: either r is adjacent to s, r is adjacent to z, w is adjacent to a new vertex, y is adjacent to a new vertex, or z is adjacent to a new vertex.

Suppose $r \sim s$. To prevent $\{sy, sz, sr\}$ from forming a claw at s, we must have $r \sim z$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Hence $s \nsim r$.

Suppose $r \sim z$. But then $\{zu, zs, zr\}$ is a claw at z since u cannot grow and $r \nsim s$. This contradicts the fact that G is claw-free and thus $r \nsim z$.

Suppose w is adjacent to an additional vertex; call it p. Recall the vertex p must be in the wrt-face. To prevent $\{wr, wv, wp\}$ from forming a claw at w, we must have $p \sim r$. To prevent $\{wt, wv, wp\}$ from forming a claw at w, we must have $p \sim t$. Note that y cannot grow into the yst-face by 3-connectivity, since $d(t) = 5$ and so t cannot grow. Hence $d(y) = 4$. Call this semi-known subgraph S. Let C be the vertex of $S - N[q]$ containing v, so that $V(C) = \{u, v, x, y, w, t, p\}$. Note that $p \nsim q$ by planarity. Then C is not well-covered since $\{v, p\}$ and $\{u, y, p\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it p. Recall the vertex p must be in the yst-face. To prevent $\{yp, ys, yx\}$ from forming a claw at y, we must have $p \sim s$. To prevent $\{yp, yt, yv\}$ from forming a claw at y, we must have $p \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y, t, p\}$. Then C is not well-covered since $\{v, p\}$ and $\{w, x, p\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex.

Suppose z is adjacent to an additional vertex; call it p. Recall the vertex p must be in the zqs-face. To prevent $\{zp, zq, zu\}$ from forming a claw at z, we must have $p \sim q$. To prevent $\{zp, zs, zu\}$ from forming a claw at z, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v. Then C is not well-covered since $\{v, p\}$ and $\{u, y, p\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Therefore the graph shown in 13(b) cannot grow, and so $q \nsim r$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yp, ys, yv\}$ from forming a claw at y, we must have $p \sim s$. To prevent $\{yt, yp, yx\}$ from forming a claw at y, we must have $p \sim t$. Thus p lies in the yst-face and $d(t) = 5 = d(s)$. This implies that $d(w) = 4$ since otherwise an additional neighbor along with u and t would form a claw at w, and similarly $d(z) = 4$ to avoid a claw at z with x and s. Since G is 3-connected, we cannot add any additional vertices or $\{r, q\}$ would be a 2-cut. But then $\{w, t\}$ is a 2-cut since $q \nsim r$. Thus y is not adjacent to an additional vertex.

Therefore w is not adjacent to an additional vertex, and so $y \nsim s$, and by symmetry, $w \nsim s$. Note that since s is adjacent to neither w nor y, we may assume that s is not adjacent to t by claw-freedom, for the rest of the proof of this claim.

Suppose y is adjacent to an additional vertex; call it r. To prevent $\{yr, yt, yv\}$ from forming a claw at y, we must have $r \sim t$. See Figure 14(a) for an illustration. Since G is 3-connected, $\{y, t\}$ is not a 2-cut, and so there must be a path from r to the vertex set $\{w, z, s\}$ that does not pass through y or t. Either r is adjacent to w, r is adjacent to z, r is adjacent to s, or r is adjacent to an additional vertex in the

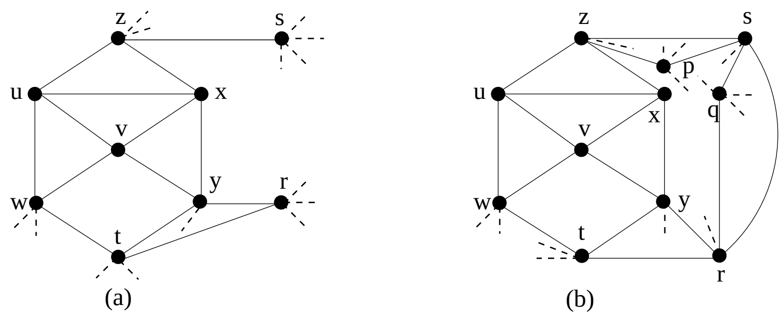


Figure 14: Proving that every vertex of degree four must lie on a K_4.

$zxyrtwu$-face.

Suppose $r \sim w$. Since G is 3-connected, $\{w, y\}$ is not a 2-cut, and so there must be a path from r to z or s that does not pass through either w or y. Either r is adjacent to z, r is adjacent to s or r is adjacent to an additional vertex in the $zxyrwu$-face. Suppose $r \sim z$. But then $\{ry, rw, rz\}$ is a claw at r, since z is not adjacent to w or y by Claim 2.4/1.3.4.1.1.2, and $w \nsim y$ by Claim 2.4/1.3.4.1.1.1. This contradicts the fact that G is claw-free. Hence $r \nsim z$. Suppose $r \sim s$. But then $\{ry, rw, rs\}$ is a claw at r, since s is not adjacent to w or y by a previous subcase (of the z adjacent to an additional vertex subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.1). This contradicts the fact that G is claw-free. Hence $r \nsim s$. So suppose that r is adjacent to an additional vertex; call it q. To prevent $\{ry, rq, rw\}$ from forming a claw at r, either y is adjacent to q or w is adjacent to q. Suppose $y \sim q$. But then $\{yt, yq, yv\}$ is a claw at y since $t \nsim q$ by planarity. Hence $y \nsim q$. Suppose $w \sim q$. But then $\{wt, wq, wv\}$ is a claw at y since $t \nsim q$ by planarity. Hence $w \nsim q$, and therefore $r \nsim w$.

Suppose $r \sim z$. To prevent $\{zu, zs, zr\}$ from forming a claw at z, we must have $r \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, w\}$ are both maximal independent sets of C. Thus by Lemma 2.2 and since

$r \nsim w$ by the preceding subcase, w must be adjacent to an additional vertex; call it q. To prevent $\{wq, wu, wt\}$ from forming a claw at w, we must have $t \sim q$. Now $z \nsim q$; otherwise $\{zq, zu, zr\}$ would be a claw at z, contradicting the fact that G is claw-free. Note that z may also not have any additional neighbors in the $zuwqs$-face; otherwise the additional neighbor along with u and r would form a claw at z, contradicting the fact that G is claw-free. Also q is not adjacent to r by birth. Call this semi-known subgraph S. Let C be the component of $S - N[s, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered, since $\{v\}$ and $\{x, y\}$ are both maximal independent sets of C. Then by Lemma 2.2 either q is adjacent to s, or y is adjacent to an additional vertex. Suppose $q \sim s$. Then again the resulting graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. However there are no possible additional edges between known vertices, and the possible faces in which additional vertices might be located are the same, by symmetry, as the possibilities when we had another graph isomorphic to the one in Figure 2(d), near the beginning of the proof of Claim 2.4/1.3.4.1.1.3.1. Hence we may assume $s \nsim q$. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yt, yp\}$ and $\{yv, yr, yp\}$ from becoming claws at y, we must have $p \sim t$ and $p \sim r$. Hence p must be in the yrt-face. Since $d(y) = d(t) = d(r) = 5$, p cannot grow and must have degree three by 3-connectivity. Call this semi-known subgraph S. Let C be the component of $S - N[z, q]$ containing y, so that $V(C) = \{v, y, p\}$. Then C is not well-covered, since $\{y\}$ and $\{v, p\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $r \nsim z$.

Suppose $r \sim s$. Since G is 3-connected, $\{z, r\}$ is not a 2-cut, and so there must be a path from s to the vertex set $\{w, y, t\}$ that does not pass through z or r. Thus since $s \nsim w$ and $s \nsim y$, either $s \sim t$, s is adjacent to an additional vertex in the $zsryx$-face, or s is adjacent to an additional vertex in the $srtwuz$-face. Suppose $s \sim t$. Then $\{ty, tw, ts\}$ is a claw at t, contradicting the fact that G is claw-free. Hence $s \nsim t$.

Suppose s is adjacent to an additional vertex in the $zsryx$-face; call it q. To prevent $\{sq, sr, sz\}$ from forming a claw at s, either $q \sim z$ or $q \sim r$. Recall that $r \nsim z$ by the preceding subcase.

Suppose $q \sim z$. Since G is 3-connected, $\{z, s\}$ is not a 2-cut and so there must be a path from q to r that does not pass through either z or s. Note that y is not adjacent to anything in the $zsryx$-face; otherwise this additional neighbor along with v and t would form a claw at y, contradicting the fact that G is claw-free. Thus either q is adjacent to r or r is adjacent to an additional vertex in the $zsryx$-face.

Suppose $q \sim r$. Now $r \nsim z$ by the preceding case, and r is not adjacent to an additional vertex in the $yxzqr$-face; otherwise this additional vertex along with s and t would form a claw at r, contradicting the fact that G is claw-free. (Recall that $s \nsim t$ by claw-freedom since $s \nsim w$ and $s \sim y$.) Note that z is not adjacent to an additional vertex in the $zsrtwu$-face; otherwise this additional vertex along with u and q would form a claw at z. Also note that r is not adjacent to a vertex in the $zsrtwu$-face; otherwise this additional vertex along with y and q will form a claw at r. Thus s is not adjacent to a vertex in the $zsrtwu$-face; otherwise this additional vertex along with z and r would form a claw at s. Hence by 3-connectivity and since u cannot grow, there are no additional vertices in the $zsrtwu$-face. But then $\{z, r\}$ is a 2-cut separating q and s from the rest of the graph, hence contradicting the fact that G is 3-connected. Hence $q \nsim r$.

Suppose r is adjacent to an additional vertex in the $zsryx$-face; call it p. To prevent $\{rt, rp, rs\}$ from forming a claw at r, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{u, v, x, w\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. By Lemma 2.2 and since $q \nsim r$, w must be adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $n \sim t$. Note that z is not adjacent to an additional vertex in the $zsrtnwu$-face; otherwise this additional vertex along with u and q would form a claw at z. Also note that r is not adjacent to a vertex in the $zsrtnwu$-face; otherwise this additional vertex along with y and p will form a claw at r. Thus s is not adjacent to a vertex in the $zsrtnwu$-face; otherwise this additional vertex, along with z and r, would form a claw at s. Hence by 3-connectivity and since u cannot grow, there are no additional vertices in the $zsrtnwu$-face. But then $\{w, t\}$ is a 2-cut separating n from the rest of the graph, and contradicting the fact that G is 3-connected. Hence $q \nsim z$.

Suppose $q \sim r$. Note that y is not adjacent to an additional vertex in the $zsqrxy$-face; otherwise the additional neighbor along with v and t would form a claw at y, contradicting the fact that G is claw-free. Since G is 3-connected, $\{r, s\}$ is not a 2-cut, and so there must be a path from q to z that does not pass through r or s. So since $q \nsim z$, z must be adjacent to an additional vertex in the $zsqrxy$-face; call it p. To prevent $\{zp, zs, zu\}$ from forming a claw at z, we must have $p \sim s$. Call this semi-known subgraph S. See Figure 14(b) for an illustration. Let C be the component of $S - N[p, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to r or w is adjacent to an additional vertex.

Suppose $p \sim r$. To prevent $\{rp, rt, rq\}$ from forming a claw at r, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to an additional vertex or w is adjacent to an additional vertex. Suppose

z is adjacent to an additional vertex; call it n. To prevent $\{zs, zn, zu\}$ from forming a claw at z, we must have $n \sim s$. To prevent $\{zp, zn, zu\}$ from forming a claw at z, we must have $n \sim p$. Thus the vertex n must be in the zps-face. But then we have the forbidden subgraph shown in Figure 5(b), and centered at p. Hence z is not adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it n. The vertex n must be in the $wuzsrt$-face. To prevent $\{wu, wt, wn\}$ from forming a claw at w, we must have $n \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[p, n]$ containing v, so that $V(C) = \{u, v, x, y\}$. Note that $p \nsim n$ by planarity. Also note that since $d(r) = 5$, r and v cannot grow, and $r \nsim v$, $d(y) = 4$. Then C cannot grow and it is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Therefore $p \nsim r$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wu, wt, wn\}$ from forming a claw at w, we must have $n \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[p, n, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex or p is adjacent to q. Suppose y is adjacent to an additional vertex; call it m. To prevent $\{yv, ym, yt\}$ from forming a claw at y, we must have $m \sim t$. To prevent $\{yv, ym, yr\}$ from forming a claw at y, we must have $m \sim r$. Thus the vertex m must be in the yrt-face. Call this semi-known subgraph S. Let C be the component of $S - N[z, n, q]$ containing v, so that $V(C) = \{v, y, m\}$. Then C is not well-covered since $\{y\}$ and $\{v, m\}$ are both maximal independent sets of C. Note that $q \nsim z$ by a previous subcase (of the s adjacent to an additional vertex in the $zsryx$-face subcase of the $r \sim s$ subcase of the y adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.1.1.3.1), $n \nsim q$ by planarity, and $z \nsim n$ by claw-freedom. Thus since C cannot grow, by Lemma 1.4, G is not well-

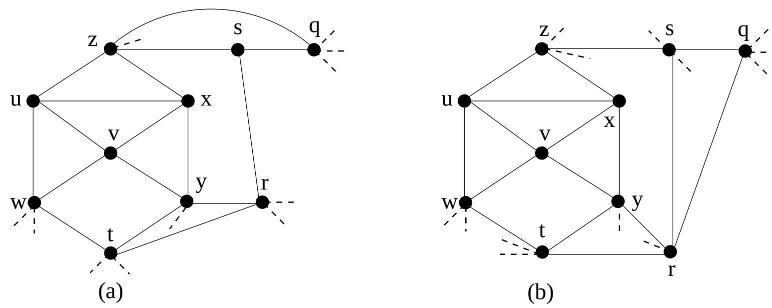


Figure 15: Proving that every vertex of degree four must lie on a K_4.

covered, a contradiction. Hence y is not adjacent to an additional vertex. Suppose $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[s, n]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2 and since C cannot grow, $n \sim s$. To prevent $\{sp, sr, sn\}$ from forming a claw at s, we must have $r \sim n$. But then $\{rq, ry, rn\}$ is a claw at r, since $d(y) = 4$, and $n \nsim q$ by planarity. This contradicts the fact that G is claw-free, so $p \nsim q$, and hence w is not adjacent to an additional vertex. Therefore $q \nsim r$, which in turn implies that s is not adjacent to an additional vertex in the $zsryx$-face.

Suppose s is adjacent to an additional vertex in the $srtwuz$-face; call it q. To prevent $\{sz, sq, sr\}$ from forming a claw at s, either $q \sim z$ or $q \sim r$.

Suppose $q \sim z$. Call this semi-known subgraph S. See Figure 15(a) for an illustration. Let C be the component of $S - N[q, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either $q \sim r$ or w is adjacent to an additional vertex.

Suppose $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing u, so that $V(C) = \{u, v, w, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to an additional vertex or w is adjacent to an additional vertex. Suppose that z is adjacent to an additional vertex; call it p. To prevent $\{zp, zs, zu\}$ and $\{zp, zq, zu\}$ from forming claws at z, we must have $s \sim p$ and $q \sim p$. Thus the vertex p must be in the zsq-face. Call this semi-known subgraph S. Let C be the component of $S - N[r, p]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $r \nsim p$ by birth. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wu, wt, wn\}$ from forming a claw at w, we must have $n \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[n, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $n \nsim s$ by planarity. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it m. To prevent $\{yv, ym, yr\}$ and $\{yv, ym, yt\}$ from forming claws at y, we must have $m \sim r$ and $m \sim t$. Now $d(y) = d(r) = d(t) = 5$, and hence by 3-connectivity, $d(m) = 3$ and m cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[n, z]$ containing y, so that $V(C) = \{v, y, m, r\}$. Then C is not well-covered since $\{y\}$ and $\{v, r\}$ are both maximal independent sets of C. Then by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $d(z) = 4$. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wu, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim s$ by planarity. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yt, yv, yn\}$ and $\{yr, yv, yn\}$ from forming claws at y, we must have $t \sim n$ and $r \sim n$. Now $d(y) = d(r) = d(t) = 5$, and hence by 3-connectivity, $d(n) = 3$ and n cannot grow. Call this semi-known subgraph S.

Let C be the component of $S - N[p, z]$ containing y, so that $V(C) = \{v, y, n, r\}$. Then C is not well-covered since $\{y\}$ and $\{v, r\}$ are both maximal independent sets of C. Then by Lemma 1.4, G is not well-covered, a contradiction. Therefore $q \nsim r$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wu, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that since $q \nsim r$ by the preceding subcase, and p is not adjacent to r or q by birth, $p \nsim s$ by claw-freedom. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yt, yv, yn\}$ and $\{yr, yv, yn\}$ from forming claws at y, we must have $t \sim n$ and $r \sim n$. Thus n must be in the yrt-face. Now $d(y) = d(t) = 5$, and hence by 3-connectivity, $d(n) = 3$ and n cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[p, z]$ containing y, so that $V(C) = \{v, y, n, r\}$. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and n are vertices of C that cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $q \nsim z$.

Suppose $q \sim r$. See Figure 15(b) for an illustration. Since G is 3-connected, $\{r, s\}$ is not a 2-cut and so there must be a path from q to the vertex set $\{z, w, t\}$ that does not pass through r and s. Thus since $q \nsim z$ by the preceding case, either q is adjacent to w, q is adjacent to t, or q is adjacent to an additional vertex in the $qrtwuzs$-face.

Suppose $q \sim w$. To prevent $\{wt, wq, wv\}$ from forming a claw at w, we must have $t \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex or z is adjacent to an additional vertex. Suppose y

is adjacent to an additional vertex; call it p. To prevent $\{yr, yp, yv\}$ and $\{yt, yp, yv\}$ from forming claws at y, we must have $p \sim r$ and $t \sim p$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at r since $t \sim q$. Thus y is not adjacent to an additional vertex and so $d(y) = 4$. Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zp, zu, zs\}$ from forming a claw at z, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing x, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are maximal independent sets of C. Note that $p \nsim q$ by birth, and C cannot grow. Hence by Lemma 1.4, G is not well-covered, a contradiction. Therefore $q \nsim w$.

Suppose $q \sim t$. Note that $q \nsim y$ by planarity. But then $\{tw, ty, tq\}$ is a claw, contradicting the fact that G is claw-free. Hence $q \nsim t$.

Suppose q is adjacent to an additional vertex in the $qrtwuzs$-face; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, q]$ containing x, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are maximal independent sets of C. Thus since $q \nsim t$ by the preceding case, by Lemma 2.2 z must be adjacent to an additional vertex; call it n. To prevent $\{zs, zn, zu\}$ from forming a claw at z, we must have $n \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[n, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to r, n is adjacent to w, or w is adjacent to an additional vertex.

Suppose $n \sim r$. But then $\{ry, rq, rn\}$ is a claw at r since y is not adjacent to either q or n by planarity and $q \nsim n$ by birth. This contradicts the fact that G is claw-free, and hence $n \nsim r$.

Suppose $n \sim w$. But then $\{wu, wt, wn\}$ is a claw at w, since u cannot grow and

$n \nsim t$ by birth. This contradicts the fact that G is claw-free and hence $n \nsim w$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wu, wt, wm\}$ from forming a claw at w, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[m, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are maximal independent sets of C. Thus by Lemma 2.2, either m is adjacent to s or y is adjacent to an additional vertex. Suppose $m \sim s$. Note that $m \nsim r$ by birth. To prevent $\{sr, sz, sm\}$ from forming a claw at s, we must have $z \sim m$. But then $\{zu, zm, zn\}$ is a claw since $m \nsim n$ by birth. This contradicts the fact that G is claw-free, and hence $m \nsim s$. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yt, yk, yv\}$ and $\{yr, yv, yk\}$ from forming claws at y, we must have $t \sim k$ and $r \sim k$. Thus k must be in the yrt-face. Now $d(y) = d(t) = d(r) = 5$, and thus by 3-connectivity, $d(k) = 3$ and k cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[s, w]$ containing y, so that $V(C) = \{x, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{x, k\}$ are maximal independent sets of C. Note that $s \nsim w$ by a previous subcase (of the z adjacent to an additional vertex subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.1) and C cannot grow. Hence by Lemma 1.4, G is not well-covered, a contradiction. Therefore w is not adjacent to an additional vertex. Hence $q \nsim r$, and so $r \nsim s$.

Suppose r is adjacent to an additional vertex in the $zxyrtwu$-face; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are maximal independent sets of C. By the preceding subcase, $r \nsim s$. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it p. To prevent $\{wt, wv, wp\}$ from forming a claw at w, we must have $t \sim p$. Note that since $s \nsim r$ by the preceding case, $s \nsim y$ by claw-freedom at y. Similarly, since $p \nsim r$ by birth, $p \nsim y$ by claw-freedom at y. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to p, q is adjacent to s, q is adjacent to y, or y is adjacent to an additional vertex.

Suppose $q \sim p$. See Figure 16(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, p]$ containing y, so that $V(C) = \{v, y, r\}$. Then C is not well-covered since $\{y\}$ and $\{v, r\}$ are maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, either y is adjacent to an additional vertex or r is adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yr, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $r \sim n$ and $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[z, p]$ containing y, so that $V(C) = \{v, y, r, n\}$. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and n are vertices of C that cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and $d(y) = 4$. Suppose r is adjacent to an additional vertex; call it n. To prevent $\{ry, rq, rn\}$ from forming a claw at r, we must have $q \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, p, s]$ containing v, so that $V(C) = \{u, x, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are maximal independent sets of C. Note that C cannot grow, $p \nsim s$ by birth and $n \nsim p$ by birth. Thus by Lemma 2.2, $n \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[n, p]$ containing x, so that $V(C) = \{u, x, v, y, z\}$. Then all the vertices of $C - x$ are adjacent to x, vertices x, u and y are vertices of C that cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $q \nsim p$.

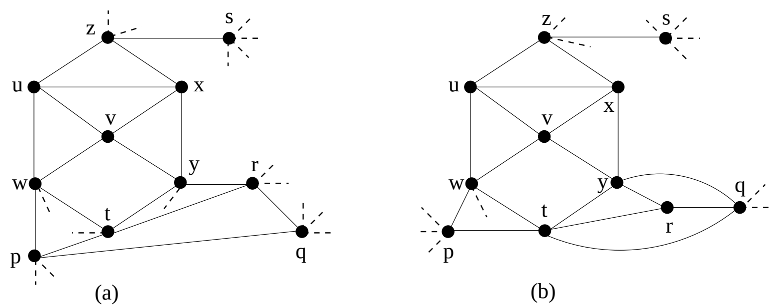


Figure 16: Proving that every vertex of degree four must lie on a K_4.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q,p]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{z, y\}$ are maximal independent sets of C. Recall $p \nsim s$ by birth. Note that $q \nsim p$ by the preceding subcase, and $z \nsim p$ otherwise $\{zp, zs, zu\}$ would be a claw at z, contradicting the fact that G is claw-free. Thus by Lemma 2.2, either z is adjacent to q, y is adjacent to q, y is adjacent to an additional vertex or z is adjacent to an additional vertex.

Suppose $z \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q,p]$ containing x, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are maximal independent sets of C. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yn, yv, yr\}$ and $\{yn, yv, yt\}$ from forming claws at y, we must have $n \sim r$ and $n \sim t$. Then n must be in the yrt-face. Note $d(y) = d(t) = 5$, and thus by 3-connectivity n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z,p]$ containing y, so that $V(C) = \{v, y, n, r\}$. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and n are vertices of C that cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $z \nsim q$.

Suppose $y \sim q$. To prevent $\{yv, yt, yq\}$ from forming a claw at y, we must have $t \sim q$. Then $d(y) = d(t) = 5$ and by 3-connectivity r cannot grow and $d(r) = 3$.

Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing x, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{z, v\}$ are maximal independent sets of C. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it n. To prevent $\{zu, zs, zn\}$ from forming a claw at z, we must have $s \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[p, z]$ containing y, so that $V(C) = \{v, y, r, q\}$. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and r are vertices of C that cannot grow, and $v \nsim r$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $y \nsim q$.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yn, yv, yr\}$ and $\{yn, yv, yt\}$ from forming claws at y, we must have $n \sim r$ and $n \sim t$. Then n must be in the yrt-face. Note $d(y) = d(t) = 5$, and thus by 3-connectivity n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, p, q]$ containing y, so that $V(C) = \{v, y, n\}$. Now C cannot grow, but it is not well-covered since $\{y\}$ and $\{v, n\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus y is not adjacent to an additional vertex and $d(y) = 4$.

Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zu, zs, zn\}$ from forming a claw at z we must have $s \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, p, q]$ containing y, so that $V(C) = \{u, v, x, y\}$. Note that n is adjacent to neither p nor q by birth. Now C cannot grow, but it is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $q \nsim s$.

Suppose $q \sim y$. To prevent $\{yq, yt, yv\}$ from forming a claw at y, we must have $t \sim q$. Note that now $d(y) = d(t) = 5$ and so by 3-connectivity, r cannot grow and so $d(r) = 3$. See Figure 16(b) for an illustration. Call this semi-known subgraph S. Let

C be the component of $S - N[z,p]$ containing y, so that $V(C) = \{v, y, r, q\}$. Recall that $p \nsim s$ by birth and so $p \nsim z$ by claw-freedom at z. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and r are vertices of C that cannot grow, and $v \nsim r$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore $y \nsim q$.

Suppose y is adjacent to an additional vertex; call it n. The vertex n may be in either the yrt-face or the exterior face. To prevent $\{yt, yv, yn\}$ and $\{yr, yv, yn\}$ from forming claws at y, we must have $t \sim n$ and $r \sim n$. If n is in the yrt-face, then by 3-connectivity and since $d(y) = d(t) = 5$, n cannot grow and thus $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z,p,q]$ containing y, so that $V(C) = \{v, y, n\}$. Recall that $p \nsim s$ by birth and so $p \nsim z$ by claw-freedom at z. Also $q \nsim p$ by a previous subcase (of the r adjacent to an additional vertex subcase of the y adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.1), and $q \nsim s$ by a previous subcase (of the r adjacent to an additional vertex subcase of the y adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.1) and so $q \nsim z$ by claw-freedom at z. Then C cannot grow, but C is not well-covered since $\{y\}$ and $\{v, n\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus n is not in the yrt-face. Suppose n is in the exterior face. Then r and q are inside the ynt-face. But since y and t cannot grow, this means $\{r, n\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence y is not adjacent to an additional vertex (a fifth neighbor).

Therefore r is not adjacent to an additional vertex and so y is not adjacent to an additional vertex (a fourth neighbor) and symmetrically w is not adjacent to an additional vertex. Hence we have proved Claim 2.4/1.3.4.1.1.3.1, and the vertex y is not adjacent to t. ∎

Claim 2.4/1.3.4.1.1.3.2: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.1.3.2: Suppose by way of contradiction, that y is adjacent to an additional vertex; call it s. By symmetry and since $y \nsim t$ by Claim 2.4/1.3.4.1.1.3.1, $w \nsim s$. Call this semi-known subgraph S. Let C be the component of $S - N[s,t]$ containing x, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to t, one of t or s is adjacent to z, or z is adjacent to an additional vertex.

Suppose $s \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to t, z is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose $z \sim t$. Recall that $w \nsim z$ by Claim 2.4/1.3.4.1.1.2 and $s \nsim w$ by Claim 2.4/1.3.4.1.1.3.1. Hence to prevent $\{tw, ts, tz\}$ from forming a claw at t, $z \sim s$. Call this semi-known subgraph S. See Figure 17(a) for an illustration. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{u, v, x, w\}$. Then C is not well-covered since $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C. Since w is not adjacent to either y or z by Claims 2.4/1.3.4.1.1.1 and 2.4/1.3.4.1.1.2 respectively, w is not adjacent to s by claw-freedom. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it r. Note that z cannot have an additional neighbor in the $zuwt$-face; otherwise this additional neighbor along with u and s would be a claw at z, contradicting the fact that G is 3-connected. Thus by 3-connectivity, the vertex r must be in the $wvyst$-face. To prevent $\{wu, wt, wr\}$ from forming a claw at w, we must have $t \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered

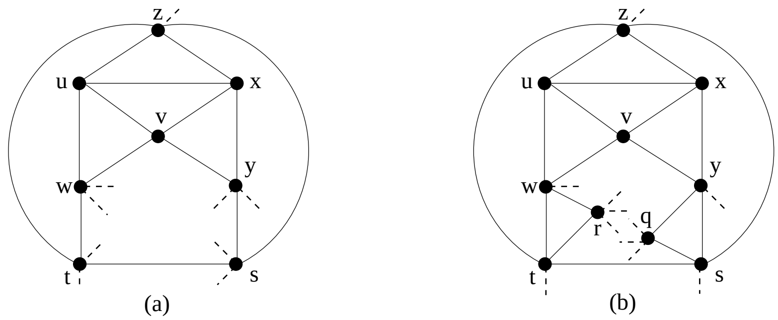


Figure 17: Proving that every vertex of degree four must lie on a K_4.

since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it q. Note that z cannot have an additional neighbor in the $zxys$-face; otherwise this additional neighbor along with u and t would be a claw at z, contradicting the fact that G is 3-connected. Thus by 3-connectivity, the vertex q must be in the $wvyst$-face. To prevent $\{yv, yq, ys\}$ from forming a claw at y, we must have $q \sim s$. Call this semi-known subgraph S. See Figure 17(b) for an illustration. Let C be the component of $S - N[r, q]$ containing x, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to q or z is adjacent to an additional vertex.

Suppose $r \sim q$. Then again the resulting graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. However there are no possible additional edges between known vertices, and the possible faces in which additional vertices might be located are the same, by symmetry, as the possibilities when we had another graph isomorphic to the one in Figure 2(d), near the beginning of the proof of Claim 2.4/1.3.4.1.1.3.1. Hence we may assume $r \nsim q$.

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zu, zt, zp\}$

and $\{zu, zs, zp\}$ from forming claws at z, we must have $t \sim p$ and $s \sim p$. Thus p must be in the zst-face by planarity. Then $d(z) = d(s) = d(t) = 5$, and so by 3-connectivity, p cannot grow and $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[w, q]$ containing z, so that $V(C) = \{x, z, p\}$. Since $q \nsim r$, $q \nsim w$ by claw-freedom at w. Then C is not well-covered since $\{z\}$ and $\{x, p\}$ are both maximal independent sets of C, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex. Thus $z \nsim t$ and similarly $z \nsim s$.

Suppose z is adjacent to an additional vertex; call it r. By birth, $r \nsim t$, and thus by symmetry we also know that $r \nsim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, t]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that since $r \nsim s$, $r \nsim y$ by claw-freedom at y. Hence by Lemma 2.2, y must be adjacent to an additional vertex; call it q. The vertex q may be in either the $vystw$-face or the exterior face. To prevent $\{yv, ys, yq\}$ from forming a claw at q, we must have $q \sim s$. One of the two possible cases is shown in Figure 18(a). Call this semi-known subgraph S. Let C be the component of $S - N[z, t]$ containing y, so that $V(C) = \{v, y, q\}$. Recall that $t \nsim q$ by birth, and since $r \nsim q$ by birth, $z \nsim q$ by claw-freedom at z. Thus by Lemma 2.2, either y is adjacent to an additional vertex or q is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it p. The vertex p may be either inside the ysq-face or exterior to that face. To prevent $\{yv, ys, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $s \sim p$ and $q \sim p$. Now $d(y) = 5$ and $d(s) = 4$. If p is inside the ysq-face, then by 3-connectivity, p cannot grow and has $d(p) = 3$. If p is exterior to that face, then q is inside the ysp-face and cannot grow by 3-connectivity. Call this semi-known subgraph S. Let C be the component of

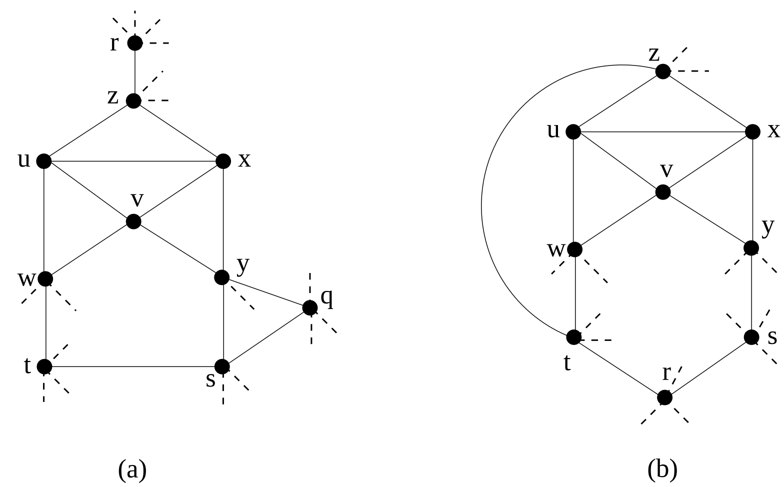


Figure 18: Proving that every vertex of degree four must lie on a K_4.

$S - N[z, t]$ containing y, so that $V(C) = \{v, y, p, q\}$. Then every vertex in $C - y$ is adjacent to y, vertices y, v and one of p or q cannot grow, and v is not adjacent to either p or q. Hence by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex (a fifth neighbor) and so $d(y) = 4$. Suppose q is adjacent to an additional vertex; call it p. By 3-connectivity, p is not in the ysq-face. Call this semi-known subgraph S. Let C by the component of $S - N[p, t, r]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets in C. This component cannot grow and $p \nsim t$ and $r \nsim t$ by birth. Thus by Lemma 2.2, p is adjacent to r. Note that this forces q and p to lie in the exterior face. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then all the vertices in $C - x$ are adjacent to x, vertices x, u and y cannot grow and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex, and thus z is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it r. The vertex r may be in the $vystw$-face or in the exterior face. To prevent $\{yr, yv, ys\}$ from forming a claw at y, we must have $r \sim s$. But then $\{u, x\}$ is a 2-cut, separating z from the rest of the graph since z is not adjacent to y, w, t, s or any additional vertices (which includes r since we added r after finding that z must not be adjacent to any additional vertices). This contradicts the fact that G is 3-connected. Hence y is not adjacent to an additional vertex, and therefore $s \nsim t$.

Suppose without loss of generality that $t \sim z$. Since G is 3-connected, $\{y\}$ is not a cut-vertex and so there must be a path from s to the vertex set $\{t, w, z\}$ that does not pass through y. Since $s \nsim t$ by the preceding case, s is not adjacent to z or w otherwise $\{zu, zs, zt\}$ or $\{wu, ws, wt\}$ would be claws respectively. Hence s must be adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[r, t]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to t, r is adjacent to y, or y is adjacent to an additional vertex.

Suppose $r \sim t$. See Figure 18(b) for an illustration. To prevent $\{tr, tw, tz\}$ from forming a claw at t, either r is adjacent to w, or r is adjacent to z. Suppose $r \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{z, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to y, r is adjacent to z, vertex z is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose $r \sim y$. Note that the edge ry may be drawn in an interior face or in the exterior face. Since G is 3-connected, $\{y, r\}$ is not a 2-cut, and so there must be a path from s to the vertex set $\{w, t, z\}$ that does not pass through y or r. So since s is not adjacent to w, t or z, s must be adjacent to an additional vertex; call it q. The vertex q may be in the ysr-face or outside of that face. Call this semi-known subgraph S. Let C be the component of $S - N[q,t]$ containing v, so that $V(C) = \{u,v,x,y\}$. Then C is not well-covered since $\{v\}$ and $\{u,y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to t, q is adjacent to y, or y is adjacent to an additional vertex.

Suppose $q \sim t$. To prevent $\{tw, tq, tz\}$ from forming a claw at t, we must have $q \sim z$ by planarity. Then again this graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. However there are no possible additional edges between known vertices, and the possible faces in which additional vertices might be located are, by symmetry, the same as the possibilities when we had another graph isomorphic to the one in Figure 2(d), near the beginning of the proof of Claim 2.4/1.3.4.1.1.3.1. Hence we may assume that $q \nsim t$.

Suppose $q \sim y$. To prevent $\{yr, yv, yq\}$ from forming a claw at y, we must have $r \sim q$. Now $d(y) = d(r) = 5$. Either s is inside the yrq-face or q is inside the ysr-face. By 3-connectivity, whichever vertex is inside cannot grow. Since G is 3-connected, $\{y,r\}$ is not a 2-cut and so there must be a path from $\{s,q\}$ to $\{z,t\}$ that does not pass through y or r. Note that since $q \nsim t$ by the preceding case, $q \nsim z$ by claw-freedom at z. Also note that if z is adjacent to an additional vertex, then t must also be adjacent to the same additional vertex; otherwise the additional vertex along with u and t would be a claw at z. A similar argument can be made to prove that if t is adjacent to an additional vertex, so is z. Since neither z nor t is adjacent to either s or q, z and t must share an additional vertex in the $yqrtzx$ or $ysrtzx$ face (depending on whether s is interior to the yrq-face or q is interior to the ysr-face); call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p,s]$ (if s is interior to the yrq-face) or $S - N[q,p]$ (if q is interior to the ysr-face) containing v, so that $V(C) = \{u,v,w,x\}$. Then C is not well-covered since $\{v\}$ and $\{w,x\}$ are both maximal independent sets of C. By planarity, the deleted set is independent. But C cannot grow since if w is adjacent to an additional vertex, then that vertex along with v and r form a claw at w. Hence by Lemma 1.4, G is not well-covered, a contradiction. Thus $q \nsim y$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yr, yv, yp\}$ and $\{ys, yv, yp\}$ from forming claws at y, we must have $r \sim p$ and $s \sim p$. Now $d(y) = d(r) = 5$. Since G is 3-connected, $\{y, r\}$ is not a 2-cut and so the vertex q must be exterior to the ysr-face, the ypr-face, and the spr-face. Thus p must be interior to the ysr-face, and so p cannot grow by 3-connectivity. Call this semi-known subgraph S. Let C be the component of $S - N[q, r]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it n. To prevent $\{zu, zt, zn\}$ from forming a claw at z, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, w, q]$ containing y, so that $V(C) = \{x, y, p\}$. Then C is not well-covered since $\{y\}$ and $\{x, p\}$ are both maximal independent sets of C. Since C cannot grow, G is not well-covered by Lemma 1.4, a contradiction. Thus y is not adjacent to an additional vertex, and therefore, $r \nsim y$.

Suppose $r \sim z$. Recall that $w \nsim z$ by Claim 2.4/1.3.4.1.1.2, and s is not adjacent to w or z by claw-freedom, since $s \nsim t$ by a previous subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2. But then $\{rw, rz, rs\}$ is a claw at r. Hence $r \nsim z$.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zu, zq, zt\}$ from forming a claw at z, we must have $q \sim t$. See Figure 19(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C. Note that $s \nsim q$; otherwise $\{sq, sy, sr\}$ would form a claw at s, contradicting the fact that G is claw-free. Also since $q \nsim r$

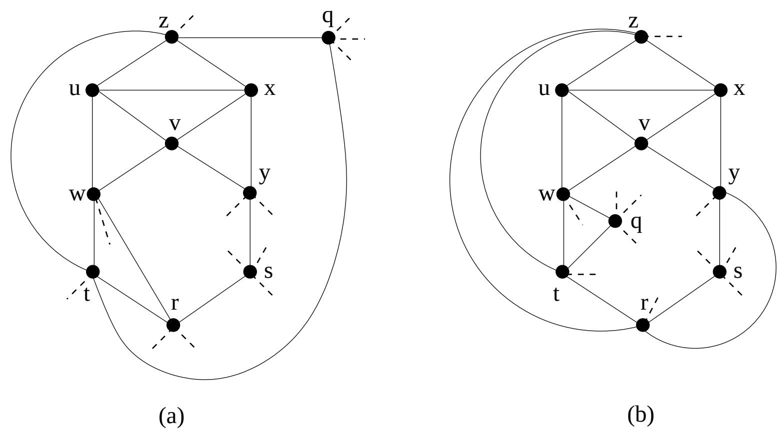


Figure 19: Proving that every vertex of degree four must lie on a K_4.

by birth, $q \nsim w$ by claw-freedom. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it p. To prevent $\{wv, wr, wp\}$ and $\{wv, wt, wp\}$ from forming claws at w, we must have $r \sim p$ and $t \sim p$. Thus p must be in the wrt-face. Note that $d(w) = d(t) = 5$, and so by 3-connectivity, p cannot grow and $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s]$ containing w, so that $V(C) = \{v, w, p\}$. Then C is not well-covered since $\{w\}$ and $\{v, p\}$ are both maximal independent sets of C. Thus since C cannot grow, by Lemma 1.4, G is not well-covered, a contradiction. Hence z must not be adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it q. Note that q could be interior or exterior to the $vysrw$-face. To prevent $\{ys, yv, yq\}$ form forming a claw at y, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q, r]$ containing u, so that $V(C) = \{u, v, x, z\}$. Note that q is not adjacent to r by birth, and by the preceding subcase, z is not adjacent to an additional vertex (and this includes q since we added q after making that observation). But C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $r \nsim w$.

Suppose $r \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to y, vertex y is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Suppose $r \sim y$. Note that the edge ry may lie in either the $wvysrt$-face or the exterior face. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it q. Note that q must be in the $wvysrt$-face by 3-connectivity since u cannot grow and z cannot have a neighbor in this face by claw-freedom. Also note that this forces the edge ry to be in the exterior face; otherwise $\{w, t\}$ is a 2-cut, separating q from the rest of the graph and contradicting the fact that G is 3-connected. To prevent $\{wt, wq, wv\}$ from forming a claw at w, we must have $t \sim q$. See Figure 19(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that $q \nsim z$ by planarity. Hence by Lemma 2.2, either q is adjacent to s, or z is adjacent to an additional vertex. Suppose $q \sim s$. Then again this graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. However there are no possible additional edges between known vertices, and the possible faces in which additional vertices might be located are symmetric with the same possibilities when we had another graph isomorphic to the one in Figure 2(d), near the beginning of the proof of Claim 2.4/1.3.4.1.1.3.1. Hence we may assume that $q \nsim s$. Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zu, zt, zp\}$ and $\{zu, zr, zp\}$ from forming claws at z, we must have $t \sim p$ and $r \sim p$, and thus p is in the ztr-face. Note that

$d(z) = d(t) = d(r) = 5$, and so by 3-connectivity, p cannot grow and $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[w, s]$ containing z, so that $V(C) = \{x, z, p\}$. Then C is not well-covered since $\{z\}$ and $\{x, p\}$ are both maximal independent sets of C. Since C cannot grow, G is not well-covered by Lemma 1.4, a contradiction. Hence z is not adjacent to an additional vertex and so $r \nsim y$.

Suppose y is adjacent to an additional vertex; call it q. Note that q may be in the $vysrtw$-face or in the exterior face. To prevent $\{ys, yq, yv\}$ from forming a claw at y, we must have $s \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $q \nsim r$ by birth. Thus by Lemma 2.2, either q is adjacent to w or w is adjacent to an additional vertex. Suppose $q \sim w$. (Note that this forces q to be in the $vysrtw$-face.) To prevent $\{wt, wq, wv\}$ from forming a claw at w, we must have $t \sim q$. Then again this graph is isomorphic to the exceptional well-covered graph in Figure 2(d). Since G is not a graph in Figure 2, this graph must grow. However there are no possible additional edges between known vertices, and the possible faces in which additional vertices might be located are symmetric with the same possibilities when we had another graph isomorphic to the one in Figure 2(d), near the beginning of the proof of Claim 2.4/1.3.4.1.1.3.1. Hence we may assume that $q \nsim w$. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wt, wp, wv\}$ from forming a claw at w, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that p is not adjacent to either q or r by birth, and so $p \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it n. To prevent $\{zn, zr, zu\}$ and $\{zn, zt, zu\}$ from forming claws at z, we must have $n \sim r$ and $n \sim t$.

Thus $d(z) = d(t) = 5$ and by 3-connectivity, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[s, w]$ containing z, so that $V(C) = \{x, z, n\}$. Then C is not well-covered since $\{z\}$ and $\{x, n\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so y is not adjacent to an additional vertex and $d(y) = 3$.

Suppose w is adjacent to an additional vertex; call it q. To prevent $\{wu, wt, wq\}$ from forming a claw at w, we must have $t \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, r]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{y, u\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex, therefore $r \nsim z$ and so $r \nsim t$.

Suppose $r \sim y$. Since G is 3-connected, $\{w, z\}$ is not a 2-cut, and so there must exist a path from t to the vertex set $\{y, s, r\}$ that does not pass through w or z. Since t is not adjacent to y (by Claim 2.4/1.3.4.1.1.3.1), s (by a previous subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), or r (by a previous subcase of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), t must be adjacent to an additional vertex; call it q. To prevent $\{tz, tw, tq\}$ from forming a claw at t, either q is adjacent to w, or q is adjacent to z.

Suppose $q \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[q, r]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that since $r \nsim t$ by the preceding subcase, $r \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, either q is adjacent to r, q is adjacent to z, or z is adjacent to an additional vertex.

Suppose $q \sim r$. Either s is in the $yrqwv$-face or s is in the exterior face.

Suppose s is in the $yrqwv$-face. Since G is 3-connected, $\{q, y\}$ is not a 2-cut, and so there must be a path from $\{z, w, t\}$ to $\{r, s\}$ that does not pass through either q or y. There are no edges between these vertex sets. The vertex w does not have any additional neighbors in the $wvysrq$-face; otherwise the additional neighbor along with t and v would form a claw at w, contradicting the fact that G is claw-free. Thus z and t must be adjacent to an additional vertex; call it p. (Note that by claw-freedom, if one of z and t is adjacent to an additional vertex, then the other must also be adjacent to that vertex.) Call this semi-known subgraph S. See Figure 20(a) for an illustration. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that since $r \nsim t$ by the preceding subcase, $r \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either p is adjacent to r or w is adjacent to an additional vertex.

Suppose $p \sim r$. To prevent $\{rq, rp, ry\}$ from forming a claw at r, either q is adjacent to y, or q is adjacent to p. (Note that since $p \nsim s$ by planarity, $p \nsim y$ by claw-freedom at y.) Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $q \sim s$. Note that this means $d(q) = 5$, and so $d(w) = 4$ otherwise if w was adjacent to an additional vertex, this additional vertex along with u and q would create a claw at w, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus $q \nsim y$. Suppose $p \sim q$. Then the vertex q does not have any additional neighbors in the $wvysrq$-face; otherwise the additional neighbor along with p and w would form a claw at q, contradicting the fact that G is claw-free. But then $\{y, r\}$ is a 2-cut, contradicting the fact that G is 3-connected. Therefore $p \nsim q$ and so $p \nsim r$.

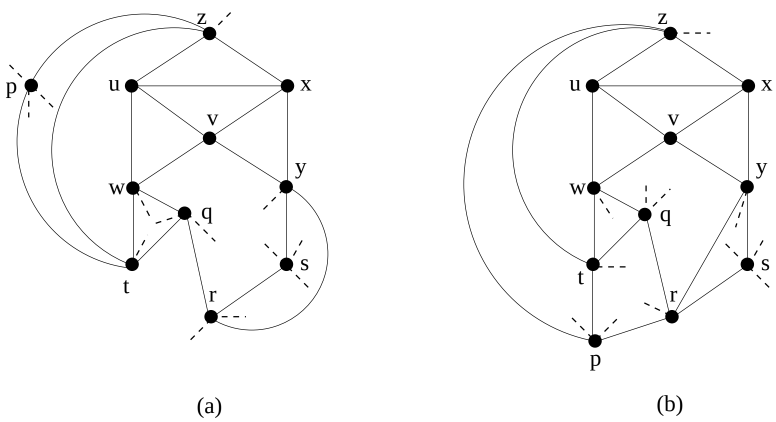


Figure 20: Proving that every vertex of degree four must lie on a K_4.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wt, wn, wv\}$ and $\{wq, wn, wv\}$ from forming claws at w, we must have $n \sim t$ and $n \sim q$. This means that n must be in the wtq-face. Note that $d(w) = d(t) = 5$, and so by 3-connectivity, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[p, y]$ containing w, so that $V(C) = \{u, w, n, q\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, u, and n cannot grow, and $u \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex, and therefore s is not in the $yrqwv$-face.

Suppose s is in the exterior face. Since G is 3-connected, $\{q, y\}$ is not a 2-cut, and so there must be a path from $\{z, w, t\}$ to $\{r, s\}$ that does not pass through either q or y. There are no edges between these vertex sets. The vertex w does not have any additional neighbors in the $wvyrq$-face; otherwise the additional neighbor along with t and v would form a claw at w, contradicting the fact that G is claw-free. Thus z and t must be adjacent to an additional vertex; call it p. (Note that by claw-freedom, if one of z and t is adjacent to an additional vertex, then the other must also be adjacent to that vertex.) Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C

is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either $p \sim r$ or w is adjacent to an additional vertex.

Suppose $p \sim r$. See Figure 20(b) for an illustration. To prevent $\{rp, rq, ry\}$ from forming a claw at r, either p is adjacent to q, or p is adjacent to y. Suppose $p \sim q$. Recall $z \nsim s$ by claw-freedom at z since $s \nsim t$. Also z does not have any additional neighbors in the $zxysrp$-face; otherwise the additional neighbor along with t and u would form a claw at z, contradicting the fact that G is claw-free. Thus $p \nsim s$ and p does not have any additional neighbors in the $zxysrp$-face; otherwise the additional neighbor along with z and q would form a claw at p, contradicting the fact that G is claw-free. But then $\{y, r\}$ is a 2-cut, disconnecting s from the rest of the graph and contradicting the fact that G is 3-connected. Hence $p \nsim q$. Suppose $p \sim y$. To prevent $\{yv, ys, yp\}$ from forming a claw at y, we must have $p \sim s$. Now $d(y) = d(p) = 5$, and so by 3-connectivity, s cannot grow and $d(s) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[q, z]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that C cannot grow. Hence by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim y$ and so $p \nsim r$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wu, wq, wn\}$ and $\{wu, wt, wn\}$ from forming claws at w, we must have $n \sim q$ and $n \sim t$. Thus n must be in the wtq-face. Now $d(w) = d(t) = 5$, and so by 3-connectivity, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[p, y]$ containing w, so that $V(C) = \{u, w, n, q\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, u, and n cannot grow, and $u \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $q \nsim r$.

Suppose $q \sim z$. Then w is not adjacent to an additional vertex in the $wvyxzq$-

face; otherwise the additional neighbor along with u and t would form a claw at w, contradicting the fact that G is claw-free. Also z is not adjacent to an additional vertex in the $wvyxzq$-face; otherwise the additional neighbor along with u and t would form a claw at z, contradicting the fact that G is claw-free. But then $\{q, y\}$ is a 2-cut, separating s and r from the rest of the graph and contradicting the fact that G is 3-connected. Hence $q \nsim z$.

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zp, zu, zt\}$ from forming a claw at z, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, w]$ containing y, so that $V(C) = \{x, y, s, r\}$. Then C is not well-covered since $\{y\}$ and $\{x, r\}$ are both maximal independent sets of C. Recall that $p \nsim q$ by birth, and so $p \nsim w$ by claw-freedom at w. Also $p \nsim r$ by birth, and so $p \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either p is adjacent to s, y is adjacent to an additional vertex, s is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing y, so that $V(C) = \{x, y, r\}$. Then C is not well-covered since $\{y\}$ and $\{x, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or r is adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yr, yn\}$ from forming claws at y, we must have $s \sim n$ and $r \sim n$. Note that $d(y) = 5$ and $d(s) = 4$, and either n is interior to the ysr-face or r is interior to the ysn-face. Thus either n or r, whichever is on the interior, cannot grow and has degree three. Then all the vertices of $C - y$ are adjacent to y, vertices y, x, and either r or n cannot grow, and x is adjacent to neither r nor n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex. Suppose r is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let

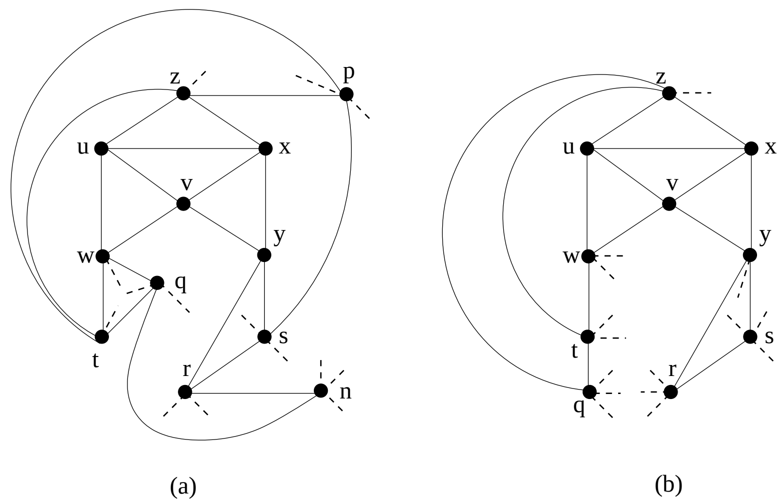


Figure 21: Proving that every vertex of degree four must lie on a K_4.

C be the component of $S - N[p, q, n]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that $p \nsim q$ by birth and $p \nsim n$ by birth. Note that C cannot grow. Thus by Lemma 2.2, n must be adjacent to q. See Figure 21(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p, n]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, u and y cannot grow, and $u \sim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $p \nsim s$.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yr, yn\}$ from forming claws at y, we must have $s \sim n$ and $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing y, so that $V(C) = \{x, y, n, r, s\}$. Either n is interior to the ysr-face, s is interior to the yrn-face, or r is interior to the ysn-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, whichever vertex of s, r or n is interior cannot grow and must have degree three. Then every vertex in $C - y$ is adjacent to y, vertices y, x, and one of s, r or n cannot grow, and x is not adjacent to any of s, r, or n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex, and hence $d(y) = 4$.

Suppose s is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, n]$ containing y, so that $V(C) = \{x, y, r\}$. Then C is not well-covered since $\{y\}$ and $\{x, r\}$ are both maximal independent sets of C. Recall that $n \nsim p$ by birth. Thus by Lemma 2.2, either r is adjacent to n, or r is adjacent to an additional vertex. Suppose $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, n must be adjacent to q. Call this semi-known subgraph S. Let C be the component of $S - N[n, p]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, y and u cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $r \nsim n$. Suppose r is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, n, m]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that m is adjacent to neither p nor q by birth. Thus by Lemma 2.2, either n is adjacent to q, or m is adjacent to q. In either case, call the semi-known subgraph S. Let C be the component of $S - N[n, m, p]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus r is not adjacent to an additional vertex (within the s adjacent to an additional vertex subcase) and so s is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, n]$ containing y, so that

$V(C) = \{x, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{x, s\}$ are both maximal independent sets of C. But C cannot grow since s is not adjacent to any additional vertices (which includes n since n was added after this statement was proved). Thus by Lemma 1.4, G is not well-covered, a contradiction. Therefore r is not adjacent to an additional vertex and so z is not adjacent to an additional vertex, and hence $q \nsim w$.

Suppose $q \sim z$. Call this semi-known subgraph S. See Figure 21(b) for an illustration. Let C be the component of $S - N[q, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that $r \nsim t$ by a previous subcase (of the $t \sim z$ subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), and so $r \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either r is adjacent to q, or w is adjacent to an additional vertex.

Suppose $r \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to s or w is adjacent to an additional vertex.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to y, vertex y is adjacent to an additional vertex, or w is adjacent to an additional vertex. Suppose $q \sim y$. Then $d(y) = d(q) = 5$, and either r is interior to the ysq-face or s is interior to the yrq-face. Whichever vertex is interior cannot grow by 3-connectivity. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2,

w is adjacent to an additional vertex; call it p. To prevent $\{wu, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, z]$ containing v, so that $V(C) = \{v, y, s, r\}$. Then all the vertices of $C - y$ are adjacent to y, vertices y, v and one of s or r cannot grow, and v is not adjacent to s or v. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim y$. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$ and $\{yv, yr, yp\}$ from forming claws at y, we must have $s \sim p$ and $r \sim p$. Thus p must be in the yrs-face. Then $d(y) = 5$ and so by Claim 2.4/1.2 p cannot grow and $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[q, w]$ containing y, so that $V(C) = \{x, y, p\}$. Then C is not well-covered since $\{y\}$ and $\{x, p\}$ are both maximal independent sets of C. But C cannot grow and so by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and $d(y) = 4$. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wu, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. Now call this semi-known subgraph S. Let C be the component of $S - N[q, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Recall that $q \nsim p$ by birth. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Then by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (in the $q \sim s$ subcase) and so $q \nsim s$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wu, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered, since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to r, or z is adjacent to an additional vertex. Suppose $p \sim r$. Note that $q \nsim s$ by the preceding subcase and $p \nsim s$ by birth, so neither p nor q is adjacent to y by claw-freedom at y. Also $p \nsim q$ by birth. But then $\{ry, rp, rq\}$ is a claw at r, contradicting the fact that G is claw-free. Hence

$p \nsim r$. Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zu, zt, zn\}$ and $\{zu, zq, zn\}$ from forming claws at z, we must have $t \sim n$ and $q \sim n$. Thus n must be in the zqt-face. Now $d(z) = d(t) = 5$ and so by 3-connectivity, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[p, y]$ containing z, so that $V(C) = \{u, z, q, n\}$. Then all the vertices of $C - z$ are adjacent to z, vertices z, u and n cannot grow, and $u \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $q \nsim r$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wp, wv, wt\}$ from forming a claw at w, we must have $p \sim t$. See Figure 22(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing w, so that $V(C) = \{v, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{v, p\}$ are both maximal independent sets of C. Recall that $p \nsim r$ by birth, and $p \nsim q$ by birth and so $p \nsim z$ by claw-freedom at z. Also $r \nsim q$ by the preceding subcase, and so $r \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, either w is adjacent to an additional vertex or p is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wn, wp\}$ and $\{wv, wn, wt\}$ from forming claws at w, we must have $n \sim p$ and $n \sim t$. Then $d(w) = d(t) = 5$, and either p is interior to the wnt-face or n is interior to the wpt-face. By Claim 2.4/1.2, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing w, so that $V(C) = \{v, w, p, n\}$. Then all the vertices of $C - w$ are adjacent to w, vertices w, v and either p or n cannot grow, and v is not adjacent to either p or n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus w is not adjacent to an additional vertex and so $d(w) = 4$.

Suppose p is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[n, q, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow, and $n \nsim r$ by birth. Thus by Lemma 2.2, n must be adjacent to q. Call this semi-known subgraph S. Let C be the component of $S - N[q, y]$ containing w, so that $V(C) = \{u, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Since $q \nsim r$ by a previous subcase (of the q adjacent to z subcase of the r adjacent to y subcase of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), $q \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it m. To prevent $\{pw, pn, pm\}$ from forming a claw at p, we must have $n \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[m, q, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{u\}$ and $\{x, w\}$ are both maximal independent sets of C. Note that $m \nsim q$ by birth, and C cannot grow. Thus by Lemma 2.2, m must be adjacent to r. Call this semi-known subgraph S. Let C be the component of $S - N[m, s, q]$ containing v, so that $V(C) = \{u, v, w, x\}$. Note that C still cannot grow. Thus by Lemma 2.2., either q is adjacent to s or m is adjacent to s. Suppose $q \sim s$. To prevent $\{qz, qn, qs\}$ from forming a claw at q, we must have $n \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[m, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, w, and x cannot grow, and $w \nsim x$. Hence by Lemma 2.3, G is not well-covered, a contradiction. Thus $q \nsim s$. Suppose $m \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[m, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, w, and x cannot grow, and $w \nsim x$. Hence by Lemma 2.3, G is not well-covered, a contradiction. Therefore $m \nsim s$ and so p is not adjacent to an additional vertex, and hence w is not adjacent to an additional vertex. Therefore $q \nsim z$ and so $r \nsim y$.

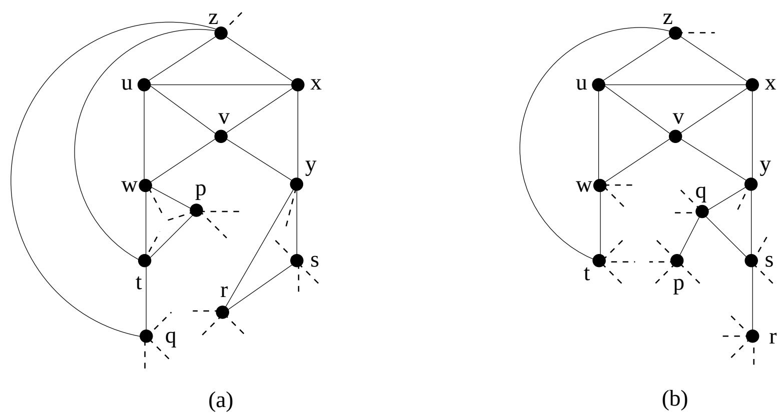


Figure 22: Proving that every vertex of degree four must lie on a K_4.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $s \sim q$. By birth, q is adjacent to neither r nor t and thus by claw-freedom, q is adjacent to neither w nor z. Since G is 3-connected, $\{y, s\}$ is not a 2-cut, and so q must be adjacent to an additional vertex; call it p. See Figure 22(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p, t, r]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that $r \nsim t$ by a previous subcase (of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2). Thus by Lemma 2.2, either p is adjacent to t, p is adjacent to r, p is adjacent to y, or y is adjacent to an additional vertex.

Suppose $p \sim t$. To prevent $\{tw, tz, tp\}$ from forming a claw at t, either $p \sim w$ or $p \sim z$.

Suppose $p \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Then by Lemma 2.2, either p is adjacent to s, or z is adjacent to an additional vertex. (Note that $p \nsim z$ or $\{pz, pw, pq\}$ would form a claw at p.) Suppose $p \sim s$. To prevent $\{sy, sr, sp\}$ from forming a claw at s, we must have $p \sim y$. (Recall that $r \nsim y$ by the preceding case. Note that $p \nsim r$ or $\{pw, pq, pr\}$ would form a claw at p.) Note that this implies that q must be interior to the ysp-face; otherwise $\{s, q\}$ would be a 2-cut, separating r from the rest of the graph and contradicting the fact that G is 3-connected. Thus now that $d(y) = d(p) = 5$, by 3-connectivity, q cannot grow and $d(q) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it n. To prevent $\{zt, zn, zu\}$ from forming a claw at z, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[r, w, n]$ containing y, so that $V(C) = \{x, y, q\}$. Then C is not well-covered, since $\{y\}$ and $\{x, q\}$ are both maximal independent sets of C. Note that C cannot grow, $n \nsim r$ by birth, and $n \nsim p$ by birth implies $n \nsim w$ by claw-freedom at w. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim s$.

Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zt, zn, zu\}$ from forming a claw at z, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C. Note that $q \nsim n$; otherwise $\{qp, qn, qs\}$ would be a claw at q, contradicting the fact that G is claw-free. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it m. To prevent $\{wm, wp, wv\}$ and $\{wm, wt, wv\}$ from forming claws at w, we must have $m \sim p$ and $t \sim m$. Thus m must be in the wpt-face. Now $d(w) = d(t) = 5$, and so by 3-connectivity, m cannot grow and $d(m) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, q]$ containing w, so that $V(C) = \{v, w, m\}$. Then C is not well-covered since $\{w\}$ and $\{v, m\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $p \nsim w$.

Suppose $p \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Then by Lemma 2.2, either p is adjacent to s, or w is adjacent to an additional vertex.

Suppose $p \sim s$. To prevent $\{sy, sp, sr\}$ from forming a claw at s, we must have $p \sim y$. (Note that $p \nsim r$; otherwise $\{pw, pq, pr\}$ would be a claw at p, contradicting the fact that G is claw-free.) Note that this implies that q must be interior to the ysp-face; otherwise $\{s, q\}$ would be a 2-cut, separating r from the rest of the graph and contradicting the fact that G is 3-connected. Thus now that $d(y) = d(p) = 5$, by 3-connectivity, q cannot grow and $d(q) = 3$. Also note that $d(z) = 4$ and z cannot grow; otherwise an additional neighbor along with u and p would form a claw at z. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wt, wn, wu\}$ from forming a claw at w, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered, since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim s$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wt, wn, wu\}$ from forming a claw at w, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Since n is not adjacent to either p or s by birth, n is not adjacent to q; otherwise $\{qp, qs, qn\}$ would be a claw at q, contradicting the fact that G is claw-free. Then by Lemma 2.2, z must be adjacent to an additional vertex; call it m. To prevent

$\{zu, zt, zm\}$ and $\{zu, zp, zm\}$ from forming claws at z, we must have $t \sim m$ and $p \sim m$. Thus by planarity, m must be in the ztp-face. Now $d(z) = d(t) = 5$, and so by 3-connectivity, m cannot grow and $d(m) = 3$. Now call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing z, so that $V(C) = \{u, z, p, m\}$. Then all the vertices of $C - z$ are adjacent to z, vertices z, u and m cannot grow, and $u \nsim m$. Thus by Lemma 2.3, G is not well-covered, and so w is not adjacent to an additional vertex and therefore $p \nsim t$.

Suppose $p \sim r$. Since $p \nsim t$ (by the preceding case), $s \nsim t$ (by a previous subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2) and $r \nsim t$ (by a previous subcase of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), and $q \nsim t$ by birth, t must be adjacent to an additional vertex in the exterior face; otherwise $\{w, z\}$ is a 2-cut, contradicting the fact that G is 3-connected. Call this additional neighbor n. See Figure 23(a) for an illustration. To prevent $\{tn, tw, tz\}$ from forming a claw at t, either n is adjacent to w or n is adjacent to z.

Suppose $n \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Note that r is not in C or $N[n, q]$. Since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), $r \nsim z$. Thus by Lemma 2.2, either n is adjacent to q, n is adjacent to z, or z is adjacent to an additional vertex.

Suppose $n \sim q$. To prevent $\{qp, qn, qs\}$ from forming a claw at q, either n is adjacent to p, n is adjacent to s, or p is adjacent to s.

Suppose $n \sim p$. Now call this semi-known subgraph S. Let C be the component of $S - N[n, s]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to s, n is adjacent to z, or z is adjacent to an additional vertex.

Suppose $n \sim s$. Note that $d(n) = 5$ and so n cannot grow. But then $\{sr, sn, sy\}$ is a claw at s, contradicting the fact that G is claw-free. Hence $n \nsim s$. Suppose $n \sim z$. To prevent $\{nw, nz, np\}$ from forming a claw at n, we must have $z \sim p$. But then $\{zu, zt, zp\}$ is a claw at z, contradicting the fact that G is claw-free. Hence $n \nsim z$. Suppose z is adjacent to an additional vertex; call it m. To prevent $\{zu, zt, zm\}$ from forming a claw at z, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[m, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Recall that $m \nsim q$ by birth. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wk, wv, wn\}$ and $\{wk, wv, wt\}$ from forming claws at w, we must have $k \sim n$ and $k \sim t$. Thus by planarity, k must be in the wtn-face. Now $d(w) = d(t) = d(n) = 5$, and thus by 3-connectivity, k cannot grow and $d(k) = 3$. Now call this semi-known subgraph S. Let C be the component of $S - N[z, q]$ containing w, so that $V(C) = \{v, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{v, k\}$ are maximal independent sets of C. But C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $n \nsim p$.

Suppose $n \sim s$. Note that $n \nsim r$ or $\{nw, nq, nr\}$ would form a claw at n contradicting the fact that G is claw-free. So to prevent $\{sy, sr, sn\}$ from forming a claw at s, we must have $n \sim y$. Then $d(n) = d(y) = 5$, and so $\{q, s\}$ is a 2-cut, separating p and r from the rest of the graph, and contradicting the fact that G is 3-connected. Thus $n \nsim s$.

Suppose $p \sim s$. To prevent $\{qy, qp, qn\}$ from forming a claw at q, we must have $p \sim y$. Now $d(y) = 5$, and so y cannot grow. Note that p cannot have any additional neighbors outside of the ypq-face; otherwise the additional exterior neighbor along with y and r would form a claw at p. But then $\{q, y\}$ is a 2-cut, separating p, s and r from the rest of the graph and contradicting the fact that G is 3-connected. Hence,

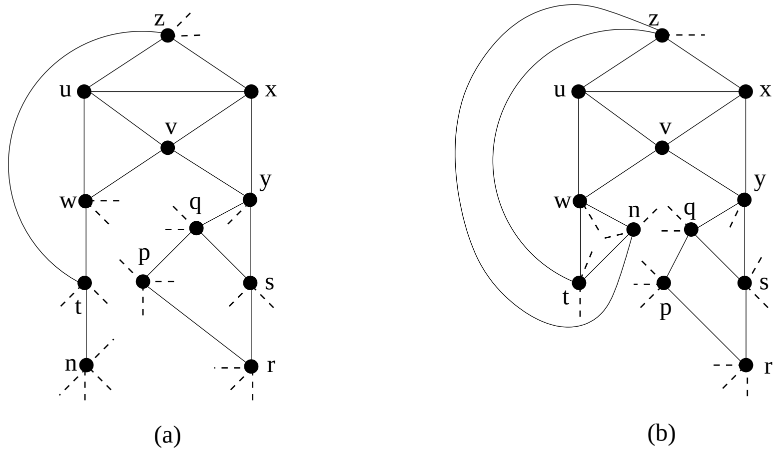


Figure 23: Proving that every vertex of degree four must lie on a K_4.

$p \nsim s$ and so $n \nsim q$.

Suppose $n \sim z$. See Figure 23(b) for an illustration. Then neither z nor w may have any additional neighbors in the exterior face by claw-freedom and so $\{n, y\}$ is a 2-cut, separating q, s, p and r from the rest of the graph and contradicting the fact that G is 3-connected. Hence $n \nsim z$.

Suppose z is adjacent to an additional vertex; call it m. To prevent $\{zm, zt, zu\}$ from forming a claw at z, we must have $m \sim t$. Call this semi-known subgraph S. Let C be a component of $S - N[m, y, r]$ containing w, so that $V(C) = \{v, w, n\}$. Then C is not well-covered since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that since m is not adjacent to q or n by birth, m is not adjacent to either w or y by claw-freedom at those vertices. Also, since $n \nsim q$ by a previous subcase (of the n adjacent to w subcase of the p adjacent to r subcase of the y adjacent to an additional vertex subcase of the t adjacent to r subcase of Claim 2.4/1.3.4.1.1.3.2), $n \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either m is adjacent to r, n is adjacent to r, w is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose $m \sim r$. To prevent $\{rp, rs, rm\}$ from forming a claw at r, either $m \sim p$ or $m \sim s$.

Suppose $m \sim p$. See Figure 24(a) for an illustration. Call this semi-known subgraph S. Let C be a component of $S - N[m, y]$ containing w, so that $V(C) = \{v, w, n\}$. Then C is not well-covered since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or n is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wn, wk\}$ and $\{wv, wt, wk\}$ from forming a claw at w, we must have $n \sim k$ and $t \sim k$. Either k is interior to the wnt-face or n is interior to the wkt-face. Since $d(w) = d(t) = 5$, by 3-connectivity, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be a component of $S - N[m, y]$ containing w, so that $V(C) = \{v, w, n, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and either n or k cannot grow, and v is not adjacent to either n or k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore, w is not adjacent to an additional vertex. Suppose n is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be a component of $S - N[m, k, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{x, w\}$ are both maximal independent sets of C. Recall that $m \nsim q$ and $m \nsim k$ by birth. Thus by Lemma 2.2, $k \sim q$. To prevent $\{qy, qp, qk\}$ from forming a claw at q, we must have $p \sim k$. Call this semi-known subgraph S. Let C be a component of $S - N[m, k, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{x, w\}$ are both maximal independent sets of C. Note that $k \nsim s$ by planarity. Thus by Lemma 2.2, $m \sim s$. To prevent $\{mz, mp, ms\}$ from forming a claw at m, we must have $p \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at p. Hence n is not adjacent to an additional vertex and so $m \nsim p$.

Suppose $m \sim s$. Call this semi-known subgraph S. Let C be a component of $S - N[m, n, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to p, or y is adjacent to an additional vertex. Suppose $n \sim p$. Call this semi-known subgraph S. Let C be a component of $S - N[m, p]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{x, w\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or y is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wu, wn, wk\}$ and $\{wu, wt, wk\}$ from forming claws at w, we must have that $n \sim k$ and $t \sim k$. Thus by planarity, k must be in the wtn-face. Now $d(w) = d(t) = 5$ and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[m, y, p]$ containing w, so that $V(C) = \{u, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Thus by Lemma 2.2, $p \sim y$. To prevent $\{ys, yv, yp\}$ from forming a claw at y, we must have $p \sim s$. Now $d(s) = 5$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Hence w is not adjacent to an additional vertex and so $d(w) = 4$. Suppose y is adjacent to an additional vertex. To prevent $\{yv, yk, ys\}$ and $\{yv, yk, yq\}$ from forming claws at y, we must have $k \sim s$ and $k \sim q$. Thus by planarity, k must be interior to the ysq-face. Now $d(y) = d(s) = 5$ and thus by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[m, p, k]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow and k is not adjacent to either m or p by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex (as a subcase of the $n \sim p$ case) and so $n \nsim p$. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv, yk, ys\}$ and $\{yv, yk, yq\}$ from forming claws at y,

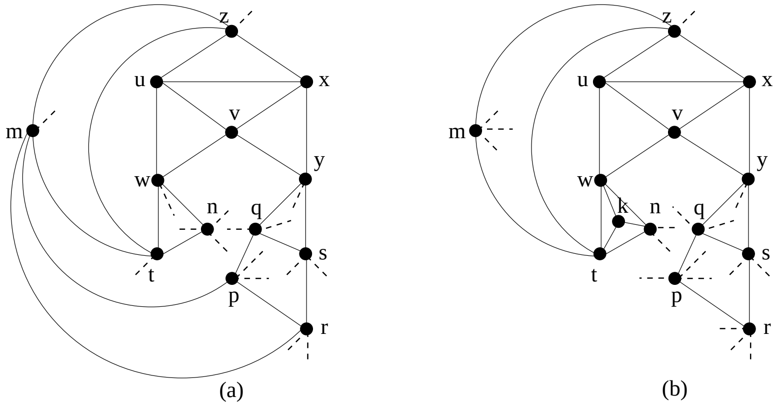


Figure 24: Proving that every vertex of degree four must lie on a K_4.

we must have $k \sim s$ and $k \sim q$. Thus by planarity, k must be interior to the ysq-face. Now $d(y) = d(s) = 5$ and thus by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[m, p, w]$ containing y, so that $V(C) = \{x, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{x, k\}$ are both maximal independent sets of C. Note that C cannot grow, and since $p \nsim t$ by a previous subcase (of the y adjacent to an additional vertex subcase of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2), $p \nsim w$ by claw-freedom at w. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $m \nsim s$, and therefore $m \nsim r$.

Suppose $n \sim r$. Note that $\{rp, rs, rn\}$ is a claw, and so either $n \sim s$ or $n \sim p$. For our arguments, it will not be necessary to distinguish these cases. Suppose we have either edge ns or edge np and call this semi-known subgraph S. Let C be the component of $S - N[m, r, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wt, wk\}$ and $\{wv, wn, wk\}$ from forming claws at w, we must have $t \sim k$ and $n \sim k$. Thus by planarity, k must be interior to the wnt-face. Now $d(w) = d(t) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[m, r, y]$ containing w, so that $V(C) = \{u, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim r$.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wt, wk\}$ and $\{wv, wn, wk\}$ from forming claws at w, we must have $t \sim k$ and $n \sim k$. Either k is interior to the wnt-face or n is interior to the wkt-face. One of the two possible cases is shown in Figure 24(b). Since $d(w) = d(t) = 5$, by 3-connectivity, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be a component of $S - N[m, y, r]$ containing w, so that $V(C) = \{v, w, n, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and either n or k cannot grow, and v is not adjacent to either n or k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus w is not adjacent to an additional vertex, and $d(w) = 4$.

Suppose n is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be a component of $S - N[k, m, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that since m and k are not adjacent to either y or r, $m \nsim s$ and $k \nsim s$ by claw-freedom at s. Then by Lemma 1.4, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so z is not adjacent to an additional vertex, and therefore $n \nsim w$.

Suppose $n \sim z$. Call this semi-known subgraph S. Let C be a component of $S - N[n, q]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that r, which is not in C or $N[n, q]$, is not adjacent to w by claw-freedom at w, since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2). Then by Lemma 2.2, either n is adjacent to q or w is adjacent to an additional vertex.

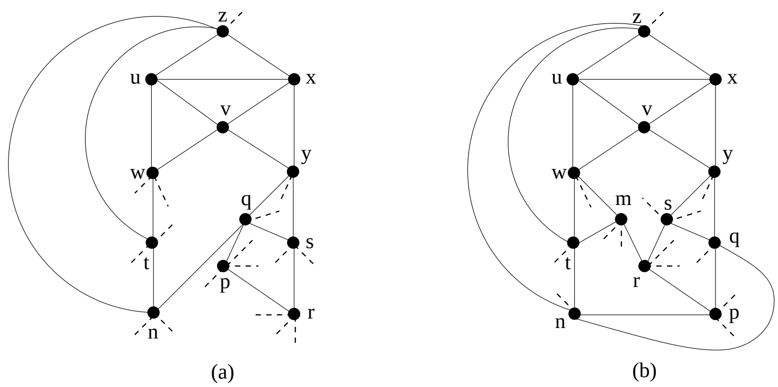


Figure 25: Proving that every vertex of degree four must lie on a K_4.

Suppose $n \sim q$. See Figure 25(a) for an illustration. To prevent $\{qy, qp, qn\}$ from forming a claw at q, either n is adjacent to y, n is adjacent to p, or p is adjacent to y.

Suppose $n \sim y$. To prevent $\{yn, yv, ys\}$ from forming a claw at y, we must have $n \sim s$. Now $d(n) = d(y) = 5$. But then $\{q, s\}$ is a 2-cut, separating p and r from the rest of the graph, and contradicting the fact that G is 3-connected. Hence $n \nsim y$.

Suppose $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[n, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to s, or w is adjacent to an additional vertex.

Suppose $n \sim s$. To prevent $\{nt, np, ns\}$ from forming a claw at n, we must have $p \sim s$, since $p \nsim t$ (by a previous subcase of the y adjacent to an additional vertex subcase of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2) and $s \nsim t$ (by a previous subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2). Now $d(n) = 5$ and thus n cannot be adjacent to any more vertices. But then $\{sy, sn, sr\}$ is a claw at s, contradicting the fact that G is claw-free. Thus $n \nsim s$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wt, wm\}$

from forming a claw at w, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[m,n,r]$ containing v, so that $V(C) = \{u,v,x,y\}$. Then C is not well-covered since $\{v\}$ and $\{u,y\}$ are both maximal independent sets of C. Note that m is not adjacent to n or s by birth, and so $m \nsim y$ by claw-freedom at y. The vertex n is not adjacent to r; otherwise $\{nr,nq,nz\}$ would be a claw at n contradicting the fact that G is claw-free. Thus by Lemma 2.2, either $m \sim r$ or y is adjacent to an additional vertex.

Suppose $m \sim r$. See Figure 25(b) for an illustration. Note that $\{rs,rp,rm\}$ is a claw at r, so either $m \sim p$ or $s \sim p$. The following arguments hold regardless of what additional edge is used. Call this semi-known subgraph S. Let C be the component of $S - N[n,r]$ containing v, so that $V(C) = \{u,v,w,x,y\}$. Then C is not well-covered since $\{v\}$ and $\{u,y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex or y is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv,wt,wk\}$ and $\{wv,wm,wk\}$ from forming claws at w, we must have $t \sim k$ and $m \sim k$. Thus by planarity, k must be interior to the wmt-face. Now $d(w) = d(t) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Now call this semi-known subgraph S. Let C be the component of $S - N[n,y,r]$ containing w, so that $V(C) = \{u,w,k\}$. Then C is not well-covered since $\{w\}$ and $\{u,k\}$ are both maximal independent sets of C. But C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv,ys,yk\}$ and $\{yv,yq,yk\}$ from forming claws at y, we must have $s \sim k$ and $q \sim k$. Thus by planarity, k must be interior to the ysq-face. Now $d(y) = d(q) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Now call this semi-known subgraph S. Let C be the component of $S - N[n,w,r]$ containing y, so that $V(C) = \{x,y,k\}$. Then C is not well-covered since $\{w\}$ and $\{x,k\}$ are both maximal independent sets of

C. But C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex (as a subcase of the $m \sim r$ case), and so $m \nsim r$.

Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv, ys, yk\}$ and $\{yv, yq, yk\}$ from forming claws at y, we must have $s \sim k$ and $q \sim k$. Thus by planarity, k must be interior to the ysq-face. Now $d(y) = d(q) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Now call this semi-known subgraph S. Let C be the component of $S - N[n, w, r]$ containing y, so that $V(C) = \{x, y, k\}$. Then C is not well-covered since $\{w\}$ and $\{x, k\}$ are both maximal independent sets of C. But C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus y is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Hence $n \nsim p$.

Suppose $p \sim y$. To prevent $\{ys, yv, yp\}$ from forming a claw at y, we must have $p \sim s$. Then p cannot be adjacent to a vertex exterior to the ypq-face; otherwise this additional neighbor along with y and r would form a claw at p, contradicting the fact that G is claw-free. But then $\{q, y\}$ is a 2-cut, separating p, s, and r from the rest of the graph and contradicting the fact that G is claw-free. Hence $p \nsim y$, and therefore $n \nsim q$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wt, wm\}$ from forming a claw at w, we must have $m \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing w, so that $V(C) = \{u, w, m\}$. Then C is not well-covered, since $\{w\}$ and $\{u, m\}$ are both maximal independent sets of C. Note that since $n \nsim q$ by the preceding subcase, $n \nsim y$ by claw-freedom at y. Also note that r and p are not in C or $N[n, y]$. Thus by Lemma 2.2, either m is adjacent to p, m is adjacent to r, w is adjacent to an additional vertex, or m is adjacent to an additional vertex.

Suppose $m \sim p$. To prevent $\{pm, pq, pr\}$ from forming a claw at p, we must have $m \sim r$. (Recall $m \nsim q$ by birth.) See Figure 26(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[n, p, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to s, p is adjacent to s, or w is adjacent to an additional vertex.

Suppose $n \sim s$. To prevent $\{sq, sr, sn\}$ from forming a claw at s, we must have $n \sim r$. Now call this semi-known subgraph S. Let C be the component of $S - N[n, p]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to y, w is adjacent to an additional vertex, or y is adjacent to an additional vertex. Suppose $p \sim y$. To prevent $\{yv, ys, yp\}$ from forming a claw at p, we must have $p \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Hence $p \nsim y$. Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wm, wk\}$ and $\{wv, wt, wk\}$ from forming claws at w, we must have $m \sim k$ and $t \sim k$. Thus by planarity, k must be in the wtm-face. Now $d(w) = d(m) = d(t) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, p]$ containing w, so that $V(C) = \{u, w, k\}$. Then C is not well-covered, since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv, yq, yk\}$ and $\{yv, ys, yk\}$ from forming claws at y, we must have $q \sim k$ and $s \sim k$. Thus by planarity, k must be in the ysq-face. Now $d(y) = d(s) = 5$ and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[k, n, p]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $n \nsim s$.

Suppose $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wm, wk\}$ and $\{wv, wt, wk\}$ from forming claws at w, we must have $m \sim k$ and $t \sim k$. Thus by planarity, k must be in the wtm-face. Now $d(w) = d(m) = d(t) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, r]$ containing w, so that $V(C) = \{u, w, k\}$. Then C is not well-covered, since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim s$.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wm, wk\}$ and $\{wv, wt, wk\}$ from forming claws at w, we must have $m \sim k$ and $t \sim k$. Thus by planarity, k must be in the wtm-face. Now $d(w) = d(m) = d(t) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, r]$ containing w, so that $V(C) = \{u, w, k\}$. Then C is not well-covered, since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Therefore w is not adjacent to an additional vertex, and so $m \nsim p$.

Suppose $m \sim r$. To prevent $\{rp, rs, rm\}$ from forming a claw at r, either m is adjacent to s or p is adjacent to s.

Suppose $m \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[m, n, p]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that

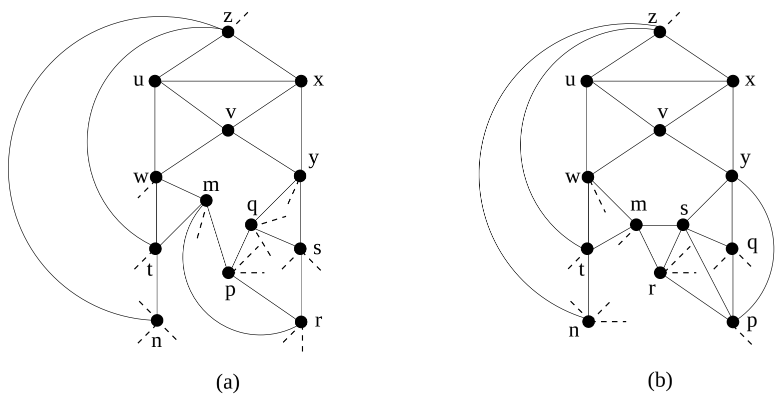


Figure 26: Proving that every vertex of degree four must lie on a K_4.

$m \nsim n$ by birth, and since $m \nsim q$ by birth, $m \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either p is adjacent to y or y is adjacent to an additional vertex. Suppose $p \sim y$. To prevent $\{yv, yp, ys\}$ from forming a claw at y, we must have $p \sim s$. See Figure 26(b) for an illustration. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Hence $p \nsim y$. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv, yq, yk\}$ and $\{yv, ys, yk\}$ from forming claws at y, we must have $q \sim k$ and $s \sim k$. Thus by planarity, k must be in the ysq-face. Now $d(y) = d(s) = 5$, and so by 3-connectivity, k cannot grow and so $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[m, z, p]$ containing y, so that $V(C) = \{v, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{v, k\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $m \nsim s$.

Suppose $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, t]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{y, u\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to y or y is adjacent to an additional vertex. Suppose $p \sim y$. Now $d(y) = 5$ and so by Claim 2.4/1.2, q cannot grow and $d(q) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[r, n, w]$ containing y, so that $V(C) = \{x, y, q\}$. Then C is not well-covered since $\{y\}$ and $\{x, q\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, n must be adjacent to r. But then $\{rp, rm, rn\}$ is a claw at r, contradicting the fact that G is claw-free. Hence $p \nsim y$. Suppose y is adjacent to an additional vertex; call it k. To prevent $\{yv, yq, yk\}$ and $\{yv, ys, yk\}$ from forming claws at y, we must have $q \sim k$ and $s \sim k$. Thus by planarity, k must be in the ysq-face. Now $d(y) = d(s) = 5$, and so by 3-connectivity, k cannot grow and so $d(k) = 3$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Hence y is not adjacent to an additional vertex and so $p \nsim s$, and therefore $m \nsim r$.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wm, wk\}$ and $\{wv, wt, wk\}$ from forming claws at w, we must have $m \sim k$ and $t \sim k$. Note that either k is interior to the wtm-face or m is interior to the wtk-face. One of the two possible cases is shown in Figure 27(a). Since $d(w) = d(t) = 5$, whichever vertex is interior cannot grow by 3-connectivity and so has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing w, so that $V(C) = \{u, w, m, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u and either k or m cannot grow, and u is adjacent to neither k or m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and $d(w) = 4$.

Suppose m is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[n, k, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $n \nsim q$ by a previous subcase (of the n adjacent to z subcase of the p adjacent to r subcase of the y adjacent to an additional vertex subcase of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2), $k \nsim n$ by birth,

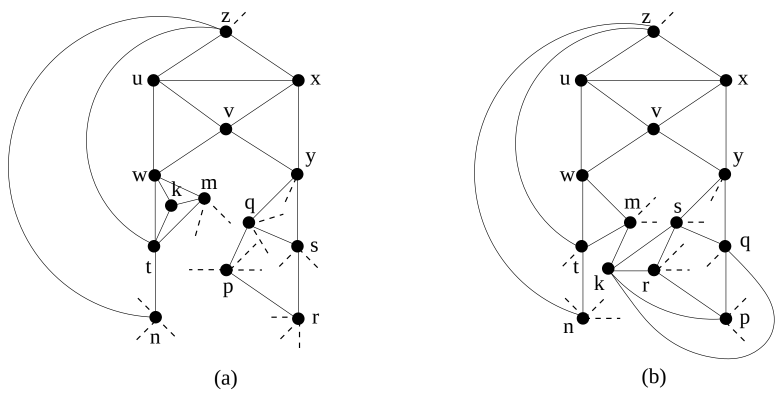


Figure 27: Proving that every vertex of degree four must lie on a K_4.

and C cannot grow. Thus by Lemma 2.2, k must be adjacent to q. To prevent $\{qy, qp, qk\}$ from forming a claw at q, either p is adjacent to y or p is adjacent to k. (Recall that $y \nsim k$ by birth.)

Suppose $p \sim y$. Then $d(y) = 5$. To prevent $\{ys, yv, yp\}$ from forming a claw at y, we must have $p \sim s$. Now if r is interior to the ypq-face, then p cannot have any neighbors exterior to the ypq-face; otherwise this additional neighbor along with y and r would form a claw at p. But then $\{q, y\}$ is a 2-cut, separating p, s and r from the rest of the graph and contradicting the fact that G is 3-connected. If r is exterior to the ypq-face, then q must be interior to the ysp-face. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, m]$ containing y, so that $V(C) = \{v, y, q\}$. Then C is not well-covered since $\{y\}$ and $\{v, q\}$ are both maximal independent sets of C. Since $d(y) = 5$, by Claim 2.4/1.2, q cannot grow and must have degree three. Thus C cannot grow. Since $m \nsim n$ by birth, $m \nsim z$ by claw-freedom at z. Note that $m \nsim r$ by a previous subcase (of the w adjacent to an additional vertex subcase of the n adjacent to z subcase of the p adjacent to r subcase of the y adjacent to an additional vertex subcase of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2), and since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase of Claim 2.4/1.3.4.1.1.3.2) $r \nsim z$ by claw-freedom at z. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim y$.

Suppose $p \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[n, k, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, either k is adjacent to s or n is adjacent to s. Suppose $k \sim s$. To prevent $\{sy, sr, sk\}$ from forming a claw at s, we must have $k \sim r$. See Figure 27(b) for an illustration. But then $\{km, kr, kq\}$ is a claw at k, since $m \nsim q$ by birth, $m \nsim r$ by planarity, and $q \nsim r$ by birth. This contradicts the fact that G is claw-free, and hence $k \nsim s$. Suppose $n \sim s$. To prevent $\{sy, sr, sn\}$ from forming a claw at s, we must have $n \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[n, k]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, w, and x, cannot grow, and $w \nsim x$. Hence by Lemma 2.3, G is not well-covered, a contradiction. So $n \nsim s$ and hence $p \nsim k$. Therefore m is not adjacent to an additional vertex and so w is not adjacent to an additional vertex, which implies that $n \nsim z$, and finally $p \nsim r$.

Suppose $p \sim y$. To prevent $\{yv, yp, ys\}$ from forming a claw at y, p must be adjacent to s. Note that either q is interior to the yps-face or p is interior to the yqs-face. One can check to see that p and q have the same adjacency relationships with the vertices of S (for example both are adjacent to y, but neither is adjacent to t). Thus at this point p and q are symmetric, and without loss of generality, we shall assume that q is interior to the yps-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, q cannot grow and must have degree three. Since G is 3-connected, $\{w, z\}$ is not a 2-cut, and so there must be a path from t to p, r and s that does not pass through w or z. Thus t must be adjacent to an additional vertex; call it n. To prevent $\{tn, tz, tw\}$ from forming a claw at t, either n is adjacent to w or n is adjacent to z.

Suppose $n \sim w$. Call this semi-known subgraph S. Let C be the component of

$S - N[n, q]$ containing u, so that $V(C) = \{u, v, x, z\}$. Note that $n \nsim q$ by planarity. Thus by Lemma 2.2, either n is adjacent to z or z is adjacent to an additional vertex. Suppose $n \sim z$. Now neither w nor z is adjacent to any vertices in the exterior face; otherwise an additional neighbor in the exterior face along with t and u would form a claw at each of z and w. But then $\{n, y\}$ is a 2-cut, separating p, q, r, and s from the rest of the graph, and contradicting the fact that G is 3-connected. Hence $n \nsim z$. Suppose z is adjacent to an additional vertex; call it m. To prevent $\{zm, zt, zu\}$ from forming a claw at z, we must have $m \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[m, w, r]$ containing y, so that $V(C) = \{x, y, p, q\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x, and q cannot grow, and $x \nsim q$. Thus if m, w and r are independent, by Lemma 2.3, G is not well-covered, which is a contradiction. Hence m, w and r must be dependent. Since $m \nsim n$ by birth, $m \nsim w$ by claw-freedom at w. Since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), $r \nsim w$ by claw-freedom at w. Hence m must be adjacent to r. See Figure 28(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[m, w]$ containing y, so that $V(C) = \{x, y, p, q, s\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x, and q cannot grow, and $x \nsim q$. Hence by Lemma 2.3, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and therefore $n \nsim w$.

Suppose $n \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing u, so that $V(C) = \{u, v, x, w\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it m. To prevent $\{wv, wt, wm\}$ from forming a claw at w, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[n, w, r]$ containing y, so that $V(C) = \{x, y, p, q\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x, and q cannot grow, and

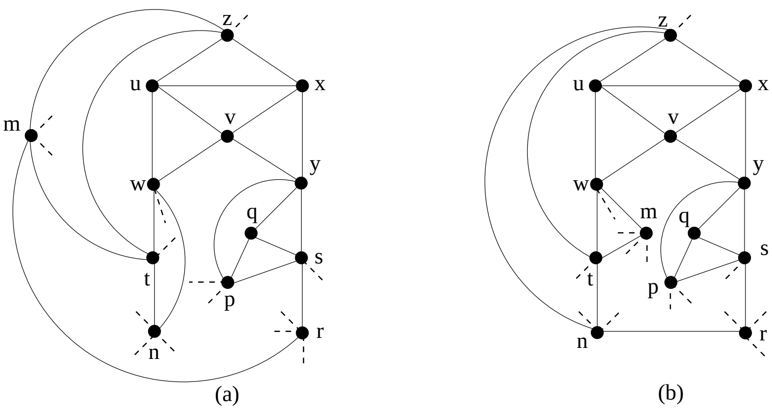


Figure 28: Proving that every vertex of degree four must lie on a K_4.

$x \nsim q$. Thus if n, w and r are independent, by Lemma 2.3, G is not well-covered, which is a contradiction. Hence n, w and r must be dependent. Since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), $r \nsim w$ by claw-freedom at w. The vertex n is not adjacent to w by the preceding case. Hence n must be adjacent to r. See Figure 28(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[n, w]$ containing y, so that $V(C) = \{x, y, p, q, s\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x, and q cannot grow, and $x \nsim q$. Hence by Lemma 2.3, G is not well-covered, a contradiction. Hence $n \nsim z$, and therefore $p \nsim y$.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yq, yn\}$ from forming claws at y, we must have $s \sim n$ and $q \sim n$. Now $d(y) = 5$. Suppose n is exterior to the yqs-face. Then either s or q is interior to either the yqn-face or the ysn-face respectively. Without loss of generality, suppose q is interior to the ysn-face. If r is interior to the ysn-face, then s cannot have any neighbors exterior to the face, or the additional exterior neighbor along with y and r would be a claw at s, contradicting the fact that G is claw-free. But then $\{n, y\}$ is a 2-cut, separating p, q, r and s from the rest of the graph and contradicting the fact that G is 3-connected. Hence r is must be exterior to the ysn-face. Then s cannot have an additional neighbor interior to the face or the additional interior neighbor along with y and r would form a claw at s, contradicting the fact that G is claw-free. But then $\{n, q\}$ is a 2-cut, separating p from the rest of the graph, and contradicting the fact that G is 3-connected. Therefore, n must be interior to the ysn-face.

Since $d(y) = 5$, and s cannot have an additional neighbor in the nqs-face without the additional neighbor forming a claw with r and y, n cannot grow by 3-connectivity and $d(n) = 3$. Since G is 3-connected, $\{w, z\}$ is not a 2-cut, and so there must be a path from t to the vertex set $\{p, q, r, s\}$ that does not pass through either w or z. Since t is not adjacent to any of the vertices in this set, t must be adjacent to an additional vertex; call it m. To prevent $\{tw, tz, tm\}$ from forming a claw at t, either m is adjacent to w or m is adjacent to z.

Suppose $m \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[m, n]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Note that $m \nsim n$ by planarity. Thus by Lemma 2.2, either m is adjacent to z or z is adjacent to an additional vertex.

Suppose $m \sim z$. See Figure 29(a) for an illustration. Then neither w nor z is adjacent to an additional vertex in the exterior face; otherwise that additional neighbor together with u and t, in both cases, creates a claw at w or z. But then $\{m, y\}$ is a 2-cut, separating the set $\{n, p, q, r, s\}$ from the rest of the graph, and contradicting the fact that G is 3-connected. Hence $m \nsim z$.

Suppose z is adjacent to an additional vertex; call it k. To prevent $\{zt, zu, zk\}$ from forming a claw at z, we must have $t \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[k, w, p, r]$ containing y, so that $V(C) = \{x, y, n\}$. Then C is not well-covered since $\{y\}$ and $\{x, n\}$ are both maximal independent sets of C. Since $k \nsim m$ by birth, $k \nsim w$ by claw-freedom at w. Recall that $p \nsim r$ by a

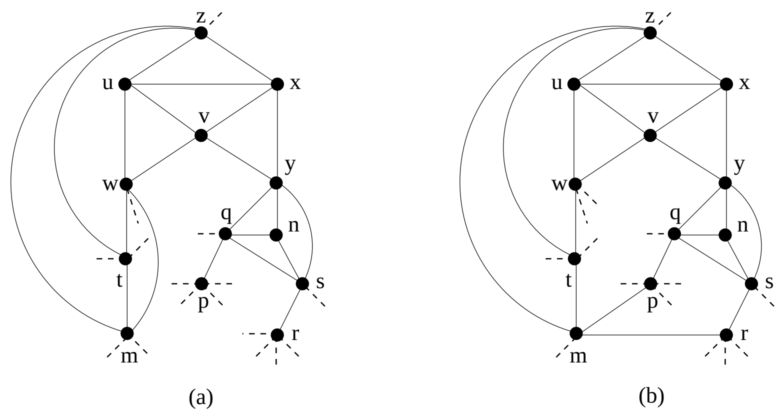


Figure 29: Proving that every vertex of degree four must lie on a K_4.

previous subcase (of the y adjacent to an additional vertex subcase of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), and note that C cannot grow. Thus by Lemma 2.2, either k is adjacent to p or k is adjacent to r. Suppose $k \sim p$. Call this semi-known subgraph S. Let C be a component of $S - N[k, w, r]$ containing y, so that $V(C) = \{x, y, n, s\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and n cannot grow, and $x \nsim n$. Thus if $\{k, w, r\}$ is independent, then by Lemma 2.3, G is not well-covered, a contradiction. Thus $\{k, w, r\}$ must be dependent and so k must be adjacent to r. But then $\{kt, kp, kr\}$ is a claw at k, contradicting the fact that G is claw-free. Thus $k \nsim p$. Suppose $k \sim r$. Call this semi-known subgraph S. Let C be a component of $S - N[k, w, p]$ containing y, so that $V(C) = \{x, y, n, s\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and n cannot grow, and $x \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $k \nsim r$, so z is not adjacent to an additional vertex, and thus $m \nsim w$.

Suppose $m \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[w, m, p, r]$ containing y, so that $V(C) = \{x, y, n\}$. Then C is not well-covered since $\{y\}$ and $\{x, n\}$ are both maximal independent sets of C. Note that

C cannot grow. Thus by Lemma 2.2, either m is adjacent to p or m is adjacent to r. Suppose $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[w, m, r]$ containing y, so that $V(C) = \{x, y, q, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and n cannot grow, and $x \nsim n$. Thus if $\{w, m, r\}$ is an independent set in G, then by Lemma 2.3, G is not well-covered, a contradiction. Thus $\{w, m, r\}$ must be dependent. Recall that $m \nsim w$ by the preceding case, and since $r \nsim t$ by a previous subcase (of the t adjacent to z subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.1.3.2), $r \nsim w$ by claw-freedom at w. Thus m is adjacent to r. See Figure 29(b) for an illustration. But then $\{mt, mp, mr\}$ is a claw at m, contradicting the fact that G is claw-free. Hence $m \nsim p$. Suppose $m \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[w, m, p]$ containing y, so that $V(C) = \{x, y, s, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and n cannot grow, and $x \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $m \nsim r$, so $m \nsim z$.

Hence y is not adjacent to an additional vertex (a fifth neighbor). Therefore, y is not adjacent to an additional vertex (a fourth neighbor), and so $t \nsim z$. By symmetry, we may also assume that $s \nsim z$.

Suppose z is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[s, w]$ containing z, so that $V(C) = \{x, z, r\}$. Then C is not well-covered since $\{z\}$ and $\{x, r\}$ are both maximal independent sets of C. Since $s \nsim t$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.1.1.3.2), $s \nsim w$ by claw-freedom at w. Note that r is not adjacent to s or t by birth, and so $r \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either z is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zu, zq, zr\}$ from forming a claw at z, we must have $q \sim r$. Call this semi-known subgraph S.

Let C be the component of $S - N[s, w]$ containing z, so that $V(C) = \{x, z, r, q\}$. Then C is not well-covered since $\{z\}$ and $\{x, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to an additional vertex, or r or q is adjacent to an additional vertex (the cases of r or q adjacent to an additional vertex are symmetric and so we will only consider one).

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zu, zq, zp\}$ and $\{zu, zr, zp\}$ from forming a claw at z, we must have $q \sim p$ and $r \sim p$. Either p is interior to the zrq-face, r is interior to the zqp-face or q is interior to the zrp-face. One of the three possible cases is shown in Figure 30(a). Now $d(z) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[s, w]$ containing z, so that $V(C) = \{x, z, r, q, n\}$. Then every vertex of $C - z$ is adjacent to z, vertices z, x and one of p, q and r cannot grow, and x is not adjacent to any of p, q or r. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus z is not adjacent to an additional vertex, as a fifth neighbor, and $d(z) = 4$.

Suppose, without loss of generality, r is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p, s, w]$ containing z, so that $V(C) = \{x, z, q\}$. Then C is not well-covered since $\{z\}$ and $\{x, q\}$ are both maximal independent sets of C. Note that p is not adjacent to either s or w by birth. Thus by Lemma 2.2, either p is adjacent to q or q is adjacent to an additional vertex. Suppose $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s, t]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, we must have $p \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, w, x, z\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, z and v

cannot grow, and $z \nsim v$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $p \nsim q$. Suppose q is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be a component of $S - N[p, n, s, t]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either t is adjacent to n or t is adjacent to p. Suppose $t \sim n$. See Figure 30(b) for an illustration. Call this semi-known subgraph S. Let C be a component of $S - N[p, n, s]$ containing u, so that $V(C) = \{u, v, w, x, z\}$. Then all the vertices of $C - u$ are adjacent to u, vertices u, z and v cannot grow, and $v \nsim z$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus $t \nsim n$. Suppose $t \sim p$. Call this semi-known subgraph S. Let C be a component of $S - N[p, n, s]$ containing u, so that $V(C) = \{u, v, w, x, z\}$. Then all the vertices of $C - u$ are adjacent to u, vertices u, z and v cannot grow, and $v \nsim z$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus $t \nsim p$, and so q is not adjacent to an additional vertex (as a subcase of r adjacent to an additional vertex), and therefore r is not adjacent to an additional vertex (as a subcase of z adjacent to an additional vertex). By symmetry we may also assume q is not adjacent to an additional vertex and so z is not adjacent to an additional vertex (as a fourth neighbor), and thus $d(z) = 3$.

Suppose r is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[q, s, t]$ containing u, so that $V(C) = \{u, v, x, z\}$. Then C is not well-covered, since $\{u\}$ and $\{z, v\}$ are both maximal independent sets of C. Note that q is not adjacent to s by birth, and C cannot grow. Thus by Lemma 2.2, q must be adjacent to t. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing u, so that $V(C) = \{u, v, w, x, z\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, z and v cannot grow, and $z \nsim v$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus r is not adjacent to an additional vertex.

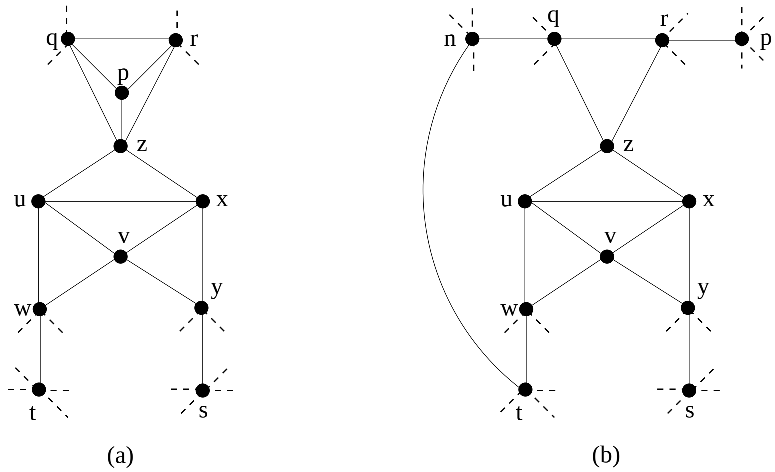


Figure 30: Proving that every vertex of degree four must lie on a K_4.

Therefore z is not adjacent to an additional vertex (a third neighbor), and hence y is not adjacent to an additional vertex. Thus we have proved Claim 2.4/1.3.4.1.1.3.2. ■

By Claim 2.4/1.3.4.1.1.3.1, we know the vertex y is not adjacent to t, and by Claim 2.4/1.3.4.1.1.3.2, we know the vertex y is not adjacent to an additional vertex. But then $\{v, x\}$ is a 2-cut, separating y from the rest of the graph, hence a contradiction. Thus we have proved Claim 2.4/1.3.4.1.1.3, and w is not adjacent to an additional vertex. ■

By Claim 2.4/1.3.4.1.1.1, $w \nsim y$. By Claim 2.4/1.3.4.1.1.2, $w \nsim z$. Furthermore by Claim 2.4/1.3.4.1.1.3, w is not adjacent to an additional vertex. But then $\{u, v\}$ is a 2-cut, separating w from the rest of the graph, hence a contradiction. Thus we have proved Claim 2.4/1.3.4.1.1, and $z \nsim u$. ■

Claim 2.4/1.3.4.1.2: The vertex z is not adjacent to y.

Proof of Claim 2.4/1.3.4.1.2: Suppose, by way of contradiction, that $z \sim y$. Either x is adjacent to an additional vertex or $d(x) = 4$.

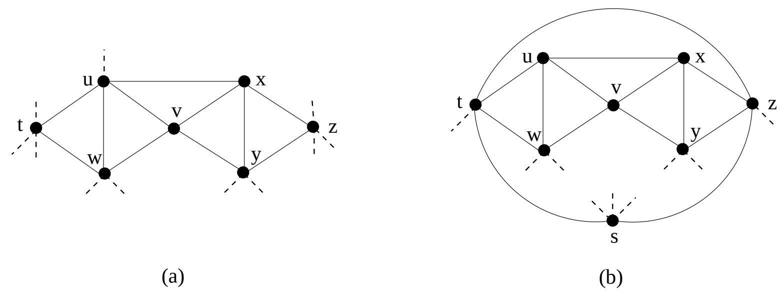


Figure 31: Proving that every vertex of degree four must lie on a K_4.

Suppose x is adjacent to an additional vertex; call it t. To prevent $\{xv, xt, xz\}$ from forming a claw at x, we must have $t \sim z$. To prevent $\{xt, xu, xy\}$ from forming a claw at x, either t is adjacent to u or t is adjacent to y. Recall $u \nsim y$ since v does not lie on a K_4. Suppose $t \sim u$. Then t must be exterior to the xyz-face, and we have the forbidden 5-wheel centered at x. Thus $t \nsim u$. Suppose $t \sim y$. Then we have the forbidden subgraph shown in Figure 5(b) and centered at x. Thus $t \nsim y$, and therefore x is not adjacent to an additional vertex.

Suppose $d(x) = 4$. Then either u is adjacent to an additional vertex or $d(u) = 3$.

Claim 2.4/1.3.4.1.2.1: The vertex u is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.2.1: Suppose, by way of contradiction, that u is adjacent to an additional vertex; call it t. To prevent $\{ux, uw, ut\}$ from forming a claw at u, we must have $w \sim t$ since x cannot grow. See Figure 31(a) for an illustration. Either u is adjacent to an additional vertex, or $d(u) = 4$.

Suppose u is adjacent to an additional vertex; call it s. To prevent $\{ut, us, uv\}$ from forming a claw at u, we must have $s \sim t$. To prevent $\{us, ux, uw\}$ from forming a claw at u, we must have $s \sim w$. Then we have the forbidden subgraph shown in Figure 5(b) and centered at u. Hence $d(u) = 4$.

Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing

x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{z, v\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to t, y is adjacent to t, z is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Claim 2.4/1.3.4.1.2.1.1: The vertex z is not adjacent to t.

Proof of Claim 2.4/1.3.4.1.2.1.1: Suppose, by way of contradiction, that $z \sim t$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(e). Since G is not a graph from Figure 2, this graph must grow. Either w is adjacent to y, z is adjacent to w (or symmetrically y is adjacent to t), or t (or symmetrically z) is adjacent to an additional vertex. Note that if w is adjacent to an additional vertex, then so is t by claw-freedom at w, and similarly for y with z.

Suppose $w \sim y$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(c). Since G is not a graph from Figure 2, this graph must grow. Either z is adjacent to w (or symmetrically y is adjacent to t), or t (or symmetrically z) is adjacent to an additional vertex. Again note that if w is adjacent to an additional vertex, then so is t by claw-freedom at w, and similarly for y with z. Suppose $w \sim z$. Then we have the forbidden 5-wheel at w. Hence $w \nsim z$, and similarly $y \nsim t$. Suppose t is adjacent to an additional vertex; call it s. To prevent $\{tu, tz, ts\}$ from forming a claw at t, we must have $s \sim z$. See Figure 31(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to w (or symmetrically s is adjacent to y), or w is adjacent to an additional vertex (or symmetrically y is adjacent to an additional vertex). Suppose $s \sim w$. To prevent $\{wu, ws, wy\}$ from forming a claw at w, we must have $s \sim y$. But then we have a forbidden 5-wheel centered at w. Hence $s \nsim w$, and similarly $s \nsim y$.

Suppose w is adjacent to an additional vertex; call it r. To prevent $\{wt, wr, wv\}$ and $\{wy, wr, wu\}$ from forming claws at w, we must have $r \sim t$ and $r \sim y$. But then again we have the forbidden 5-wheel centered at w. Hence w (and similarly y) is not adjacent to an additional vertex, and so t is not adjacent to an additional vertex. Therefore $w \nsim y$.

Suppose $z \sim w$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(f). Since G is not a graph from Figure 2, this graph must grow. Note that if w is adjacent to an additional vertex, then so is t by claw-freedom at w, and similarly for y with z. Thus either t or z is adjacent to an additional vertex. But note that if t is adjacent to an additional vertex, then z must also be adjacent to that vertex; otherwise the additional neighbor along with z and u is a claw at t, contradicting the fact that G is claw-free. Similarly, if z is adjacent to an additional vertex, then t must be adjacent to that vertex as well. Thus t and z share an additional neighbor, call it s. To prevent $\{zx, zw, zs\}$ from forming a claw at z, we must have $w \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus $z \nsim w$, and similarly $y \nsim t$.

Suppose t is adjacent to an additional vertex; call it s. To prevent $\{tu, tz, ts\}$ from forming a claw at t, we must have $z \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to w (or symmetrically y is adjacent to s), or w (or symmetrically y) is adjacent to an additional vertex.

Suppose $s \sim w$. See Figure 32(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to y , or y is adjacent to an additional vertex. Suppose $s \sim y$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(g). Since G is not a graph from Figure 2, this graph must grow. There are no possible additional adjacencies between known vertices, and so there must exist an additional vertex. By 3-connectivity, the additional vertex cannot be inside any of the faces that have either u or x or both vertices on the boundary since u and x cannot grow. If there exists an additional vertex in the tws-face, then t must have a neighbor in that face by 3-connectivity. But then this additional interior neighbor, along with u and z, forms a claw at t, contradicting the fact that G is claw-free. Hence there are no additional vertices in the tws-face. If there exists an additional vertex in the $wvys$-face, then w must have a neighbor in that face by 3-connectivity. But then this additional interior neighbor, along with t and v, forms a claw at w, contradicting the fact that G is claw-free. Hence there are no additional vertices in the $wvys$-face. If there exists an additional vertex in the zys-face, then z must have a neighbor in that face by 3-connectivity. But then this additional interior neighbor along with t and x forms a claw at z, contradicting the fact that G is claw-free. Hence there are no additional vertices in the zys-face. If there exists an additional vertex in the zts-face, then s must have a neighbor in that face by 3-connectivity. But then this additional interior neighbor along with w and y forms a claw at s, contradicting the fact that G is claw-free. Hence there are no additional vertices in the zts-face. But then G cannot grow, and so $y \nsim s$. Suppose y is adjacent to an additional vertex; call it r. To prevent $\{yv, yr, yz\}$ from forming a claw at y, we must have $r \sim z$. But then $\{zx, zr, zs\}$ is a claw at z, since $r \nsim s$ by birth. This contradicts the fact that G is claw-free and so y is not adjacent to an additional vertex. Therefore $s \nsim w$, and similarly $s \nsim y$.

Suppose, without loss of generality, that w is adjacent to an additional vertex; call it r. To prevent $\{wv, wt, wr\}$ from forming a claw at w, we must have $r \sim t$. But then $\{tu, tr, ts\}$ is a claw at t, since $r \nsim s$ by birth. This contradicts the fact

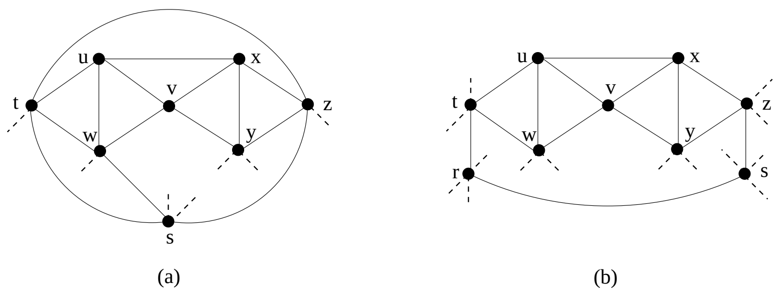


Figure 32: Proving that every vertex of degree four must lie on a K_4.

that G is claw-free, and hence w is not adjacent to an additional vertex. Similarly, y is not adjacent to an additional vertex and so t is not adjacent to an additional vertex. Similarly, z is not adjacent to an additional vertex. Therefore $z \nsim t$. ■

Claim 2.4/1.3.4.1.2.1.2: The vertex y is not adjacent to t.

Proof of Claim 2.4/1.3.4.1.2.1.2: Suppose, by way of contradiction, that $y \sim t$. But then $\{yv, yt, yz\}$ forms a claw at y, contradicting the fact that G is claw-free. Hence $y \nsim t$. ■

Claim 2.4/1.3.4.1.2.1.3: The vertex z is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.2.1.3: Suppose, by way of contradiction, that z is adjacent to an additional vertex; call it s. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing u, so that $V(C) = \{u, v, w, t\}$. Then C is not well-covered since $\{u\}$ and $\{v, t\}$ are both maximal independent sets of C. Recall $z \nsim t$ by Claim 2.4/1.3.4.1.2.1.1. Note that since $w \nsim s$ by claw-freedom at w since $s \nsim t$ by birth, $w \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, either t is adjacent to an additional vertex or w is adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal

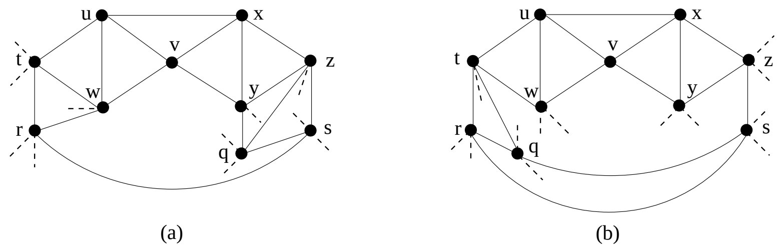


Figure 33: Proving that every vertex of degree four must lie on a K_4.

independent sets of C. Since $r \nsim z$ by birth, $r \nsim y$ by claw-freedom at y. Since $s \nsim t$ by birth, $s \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either r is adjacent to s, r is adjacent to w (or symmetrically s is adjacent to y), or w (or symmetrically y) is adjacent to an additional vertex.

Suppose $r \sim s$. See Figure 32(b) for an illustration. Since G is 3-connected, $\{s, t\}$ is not a 2-cut, and so there must be a path from r to w or y. Thus either r is adjacent to w, r is adjacent to an additional vertex in the $rtwvyzs$-face, or r is adjacent to an additional vertex in the $tuxzsr$-face.

Suppose $r \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, yz, yq\}$ from forming a claw at y, we must have $z \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing u, so that $V(C) = \{t, u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{t, v\}$ are both maximal independent sets of C. Note that since $q \nsim r$ by birth, q is not adjacent to w or t by claw-freedom at those vertices. Thus by Lemma 2.2, either q is adjacent to s, w is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose $q \sim s$. See Figure 33(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing w, so that $V(C) = \{u, v, w, r, t\}$. Then C is not well-covered since $\{w\}$ and $\{v, t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, t is adjacent to an additional vertex, or r is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wt, wv, wp\}$ and $\{wr, wv, wp\}$ from forming claws at w, we must have $p \sim t$ and $p \sim r$. Thus p must be interior to the wtr-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus w is not adjacent to an additional vertex. Suppose t is adjacent to an additional vertex; call it p. To prevent $\{tu, tr, tp\}$ from forming a claw at t, we must have $r \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by planarity, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus t is not adjacent to an additional vertex. Suppose r is adjacent to an additional vertex; call it p. To prevent $\{rt, rp, rs\}$ from forming a claw at r, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, z]$ containing u, so that $V(C) = \{u, v, w, t\}$. Then C is not well-covered since $\{u\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that $p \nsim z$ by birth and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $q \nsim s$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wt, wv, wp\}$ and $\{wr, wv, wp\}$ from forming claws at w, we must have $p \sim t$ and $p \sim r$. Thus p must be interior to the wtr-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus w is not adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it p. To prevent $\{tu, tr, tp\}$

from forming a claw at t, we must have $r \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p,q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by planarity, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus t is not adjacent to an additional vertex and so y is not adjacent to an additional vertex.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zx, zs, zq\}$ from forming a claw at z, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q,r]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $q \nsim r$ by birth, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so $r \nsim w$. By a similar argument, we may also assume that $s \nsim y$.

Suppose r is adjacent to an additional vertex in the $rtwvyzs$-face; call it q. To prevent $\{rt, rq, rs\}$ from forming a claw at r, either q is adjacent to s or q is adjacent to t. Recall that $s \nsim t$ by birth.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q,t]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to t, q is adjacent to y, q is adjacent to z, or z is adjacent to an additional vertex. (Note that if y is adjacent to an additional vertex, then z must be adjacent to that vertex as well by claw-freedom at y.)

Suppose $q \sim t$. See Figure 33(b) for an illustration. Since G is 3-connected, $\{t, s\}$ is not a 2-cut, and so there must be a path from q and r to the vertex set $\{w, y, z\}$ that does not pass through either t or s. Thus either q is adjacent to w, q is adjacent to y, q is adjacent to z, q is adjacent to an additional vertex in the $twvyzsq$-face, or

r is adjacent to an additional vertex in the $tuxzsr$-face.

Suppose $q \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to y, q is adjacent to z or z is adjacent to an additional vertex. Suppose $q \sim y$. To prevent $\{qw, qy, qr\}$ from forming a claw at q, we must have $w \sim y$. But then we have the forbidden 5-wheel subgraph centered at w. Hence $q \nsim y$. Suppose $q \sim z$. To prevent $\{qr, qw, qz\}$ from forming a claw at q, we must have $w \sim z$. But then $\{wt, wv, wz\}$ is a claw at w since $t \nsim z$ by Claim 2.4/1.3.4.1.2.1.1, and v cannot grow. This contradicts the fact that G is claw-free, and therefore $q \nsim z$. Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zx, zs, zp\}$ from forming a claw at z, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth. Thus by Lemma 2.2, either p is adjacent to y, or y is adjacent to an additional vertex. Suppose $p \sim y$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim r$ by planarity, and since $p \nsim t$ by planarity, $p \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ and $\{wv, wq, wn\}$ from forming claws at w, we must have $t \sim n$ and $q \sim n$. Thus n must be interior to the wtq-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus $p \nsim y$. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $z \sim n$. But then $\{zx, zn, zp\}$ is a claw at z, since $n \nsim p$ by birth. Thus y is not adjacent to an additional vertex and so z is not adjacent to an additional vertex, and therefore $q \nsim w$.

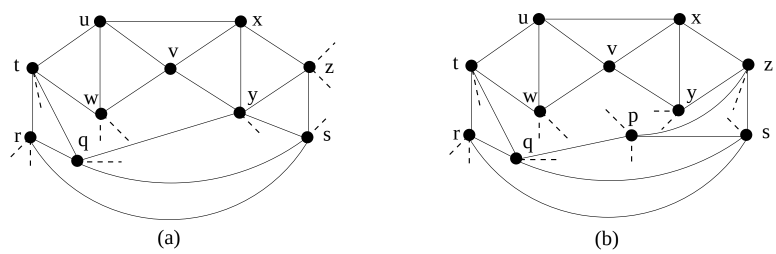


Figure 34: Proving that every vertex of degree four must lie on a K_4.

Suppose $q \sim y$. To prevent $\{qt, qy, qs\}$ from forming a claw at q, we must have $y \sim s$. See Figure 34(a) for an illustration. But $\{yv, yz, yq\}$ is a claw at y, since $q \nsim z$ by planarity. Hence $q \nsim y$.

Suppose $q \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that q is not adjacent to w or y by the two preceding subcases. Thus by Lemma 2.2, w (or symmetrically y) is adjacent to an additional vertex; call it p. To prevent $\{wt, wv, wp\}$ from forming a claw at w, we must have $t \sim p$. But then $\{tu, tp, tr\}$ is a claw at t since $p \nsim r$ by planarity and u cannot grow. Hence w (and similarly y) is not adjacent to an additional vertex and so $q \nsim z$.

Suppose q is adjacent to an additional vertex in the $twvyzsq$-face; call it p. To prevent $\{qt, qp, qs\}$ from forming a claw at q, we must have $p \sim s$. (Note that $p \nsim t$; otherwise $\{tu, tr, tp\}$ would form a claw at t contradicting the fact that G is claw-free.) Call this semi-known subgraph S. Let C be the component of $S - N[p, t]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to y, p is adjacent to z, or z is adjacent to an additional vertex. (Note that if y is adjacent to an additional vertex, then z must be adjacent to that vertex as well by claw-freedom at y.)

Suppose $p \sim y$. To prevent $\{yv, yz, yp\}$ from forming a claw at y, we must have $z \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to w, or w is adjacent to an additional vertex. Suppose $p \sim w$. But then $\{wt, wv, wp\}$ is a claw at w, since v cannot grow and $p \nsim t$. This is a contradiction and thus $p \nsim w$. Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tr, tn\}$ is a claw at t, since $r \nsim n$ by birth and planarity, and u cannot grow. This is a contradiction and thus w is not adjacent to an additional vertex and so $p \nsim y$.

Suppose $p \sim z$. See Figure 34(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim r$ by planarity. Thus by Lemma 2.2, either p is adjacent to w, vertex w is adjacent to an additional vertex, or y is adjacent to an additional vertex. Suppose $p \sim w$. To prevent $\{pw, pq, pz\}$ from forming a claw at p, we must have $w \sim z$. (Recall that q is not adjacent to w or z by previous subcases of the q adjacent to t subcase of the q adjacent to s subcase of the r adjacent to an additional vertex in the $rtwvyzs$-face subcase of the r adjacent to s subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.1.2.1.3.) But then $\{zx, zw, zs\}$ is a claw at z, contradicting the fact that G is claw-free. Hence $p \nsim w$. Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tr, tn\}$ is a claw at t, since $r \nsim n$ by planarity and u cannot grow. Thus w is not adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $z \sim n$. But then $\{zx, zn, zp\}$ is a claw at z, since $n \nsim p$ by birth and x cannot grow. Hence y is not adjacent to an additional vertex and therefore $p \nsim z$. Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zx, zn, zs\}$ from forming a claw at z, we must have $n \sim s$. But then $\{sz, sp, sr\}$ is a claw at s, since $p \nsim r$ by planarity, $p \nsim z$ by the preceding subcase, and $r \nsim z$ by birth. This contradicts the fact that G is claw-free, and hence z is not adjacent to an additional vertex, and therefore q is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex in the $tuxzsr$-face; call it p. Note that if $p \sim t$, then $\{tp, tq, tu\}$ is a claw at t since $p \nsim q$ by planarity and u cannot grow. Thus to prevent $\{rt, rs, rp\}$ from forming a claw at r, we must have $p \sim s$. (Recall that $t \nsim s$ by birth.) To prevent $\{sz, sq, sp\}$ from forming a claw at s, we must have $p \sim z$, since $p \nsim q$ by planarity, and $q \nsim z$ by a previous subcase (of the q adjacent to t subcase of the q adjacent to s subcase of the r adjacent to an additional vertex in the $rtwvyzs$-face subcase of the r adjacent to s subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.1.2.1.3.). Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex or y is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tn, tq\}$ is a claw at t, since $n \nsim q$ by birth and u cannot grow. This is a contradiction and thus w is not adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $n \sim z$. But then $\{zn, zx, zp\}$ is a claw at z, since $n \nsim p$ by birth and x cannot grow. Hence y is not adjacent to an additional vertex and so r is not adjacent to an additional vertex in the $tuxzsr$-face, and therefore $\{t, s\}$ is a 2-cut separating q and r from the rest of the graph, a contradiction. Thus $q \nsim t$.

Suppose $q \sim y$. To prevent $\{yv, yz, yq\}$ from forming a claw at y, we must have $z \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing u, so that $V(C) = \{u, v, w, x, t\}$. Then C is not well-covered since $\{u\}$ and $\{t, x\}$ are both maximal independent sets of C. Note that since $q \nsim t$ by the preceding subcase, $q \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it p. (Note that if w is adjacent to an additional vertex, then t must be adjacent to that vertex as well by claw-freedom at w.) To prevent $\{tu, tr, tp\}$ from forming a claw at t, we must have $r \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth. Thus by Lemma 2.2, either $p \sim w$ or w is adjacent to an additional vertex. Suppose $p \sim w$. See Figure 35(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing x, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim s$ by planarity and $p \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $z \sim n$. But then $\{zx, zn, zs\}$ is a claw at z, contradicting the fact that G is claw-free. Hence $p \nsim w$. Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wt, wn, wv\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tn, tp\}$ is a claw at t, since $n \nsim p$ by birth. Thus w is not adjacent to an additional vertex and therefore $q \nsim y$.

Suppose $q \sim z$. Since G is 3-connected, $\{t, z\}$ is not a 2-cut and so there must be a path from the vertex set $\{w, y\}$ to the vertex set $\{r, q, s\}$ that passes through neither t nor z. Since there are no possible edges between these vertex sets, either w is adjacent to an additional vertex, or y is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv, wt, wp\}$ from forming

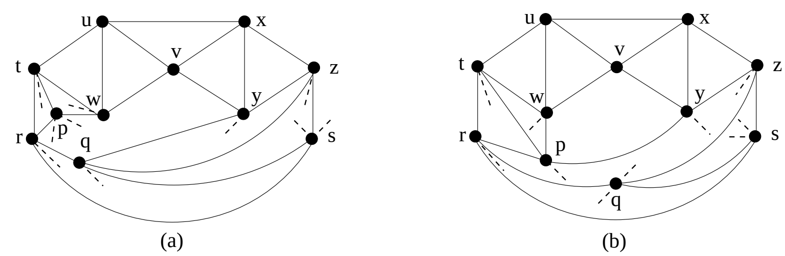


Figure 35: Proving that every vertex of degree four must lie on a K_4.

a claw at w, we must have $p \sim t$. To prevent $\{tu, tr, tp\}$ from forming a claw at t, we must have $p \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to y, or y is adjacent to an additional vertex. Suppose $p \sim y$. See Figure 35(b) for an illustration. To prevent $\{yv, yp, yz\}$ from forming a claw at y, we must have $p \sim z$. But then $\{zx, zs, zp\}$ is a claw at z, since x cannot grow and $p \nsim s$ by planarity. This is a contradiction and hence $p \nsim y$. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $z \sim n$. But then $\{zx, zn, zs\}$ is a claw at z, since $n \nsim s$ by birth. This contradicts the fact that G is claw-free. Thus y is not adjacent to an additional vertex (as a subcase of w being adjacent to an additional vertex) and therefore w is not adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yz, yp\}$ from forming a claw at y, we must have $z \sim p$. But then $\{zx, zp, zs\}$ is a claw at z, since $p \nsim s$ by planarity. This contradicts the fact that G is claw-free. Thus y is not adjacent to an additional vertex, and therefore $q \nsim z$.

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zx, zp, zs\}$ from forming a claw at z, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$.

Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that since p is not adjacent to either q or t by birth, and since $q \nsim t$ by a previous subcase (of the q adjacent to s subcase of the r adjacent to an additional vertex in the $rtwvyzs$-face subcase of the r adjacent to s subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.1.2.1.3), $p \nsim r$ otherwise $\{rt, rq, rp\}$ would be a claw at r. Thus by Lemma 2.2, either p is adjacent to y, vertex y is adjacent to an additional vertex, or w is adjacent to an additional vertex. Suppose $p \sim y$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have w adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tr, tn\}$ is a claw at t since u cannot grow and $r \nsim n$ by birth. This contradicts the fact that G is claw-free and hence $p \nsim y$. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ from forming a claw at y, we must have $z \sim n$. But then $\{zx, zn, zp\}$ is a claw at z since $n \nsim p$ by birth. This contradicts the fact that G is claw-free, and thus y is not adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wt, wn\}$ from forming a claw at w, we must have $t \sim n$. But then $\{tu, tr, tn\}$ is a claw at t since u cannot grow and $r \nsim n$ by birth. Thus w is not adjacent to an additional vertex, and therefore z is not adjacent to an additional vertex and so $q \nsim s$.

Suppose $q \sim t$. See Figure 36(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that $q \nsim s$ by the preceding subcase. Thus by Lemma 2.2, either q is adjacent to w, q is adjacent to y, w is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose $q \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[q,s]$ containing v, so that $V(C) = \{u,v,x,y\}$. Then C is not well-covered, since $\{v\}$ and $\{u,y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to y, or y is adjacent to an additional vertex. Suppose $q \sim y$. To prevent $\{yv,yz,yq\}$ from forming a claw at y, we must have $z \sim q$. But then $\{zx,zs,zq\}$ is a claw at z, contradicting the fact that G is claw-free. Hence $q \nsim y$. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv,yz,yp\}$ from forming a claw at y, we must have $z \sim p$. But then $\{zx,zs,zp\}$ is a claw at z, since $s \nsim p$ by birth. This contradicts the fact that G is claw-free, and hence y is not adjacent to an additional vertex and so $q \nsim w$.

Suppose $q \sim y$. To prevent $\{yv,yz,yq\}$ from forming a claw at y, we must have $z \sim q$. But then $\{zx,zs,zq\}$ is a claw at z, contradicting the fact that G is claw-free. Hence $q \nsim y$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv,wt,wp\}$ from forming a claw at w, we must have $t \sim p$. But then $\{tu,tp,tq\}$ is a claw, since $p \nsim q$ by birth. This contradicts the fact that G is claw-free, and hence w is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv,yz,yp\}$ from forming a claw at y, we must have $z \sim p$. But then $\{zx,zs,zp\}$ is a claw at z, since $s \nsim p$ by birth. This contradicts the fact that G is claw-free, and hence y is not adjacent to an additional vertex and so $q \nsim t$. Therefore r is not adjacent to an additional vertex in the $rtwvyzs$-face.

Since r is not adjacent to an additional vertex in the $rtwvyzs$-face, we may assume by symmetry that s is not adjacent to an additional vertex in the $rtwvyzs$-face. Then t cannot have an additional neighbor in the $rtwvyzs$-face, or this additional neighbor together with r and u would form a claw at t, contradicting the fact that G is claw-

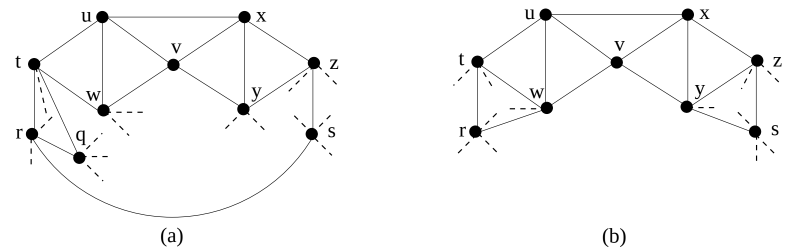


Figure 36: Proving that every vertex of degree four must lie on a K_4.

free. Similarly, z cannot have an additional neighbor in the $rtwvyzs$-face, or this additional neighbor together with s and x would form a claw at z, contradicting the fact that G is claw-free. Hence since G is 3-connected, there are no additional vertices in the $rtwvyzs$-face. But then $\{t, z\}$ is a 2-cut, separating the vertex set $\{u, v, w, x, y\}$ from the rest of the graph, and contradicting the fact that G is 3-connected. Hence the case where r is not adjacent to an additional vertex in the $rtwvyzs$-face, but adjacent to an additional vertex in the $tuxzsr$-face is not possible. Therefore $r \nsim s$.

Suppose $r \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that since $r \nsim z$ by birth, $r \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either s is adjacent to y or y is adjacent to an additional vertex.

Suppose $s \sim y$. See Figure 36(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing t, so that $V(C) = \{u, t, w, r\}$. Then C is not well-covered, since $\{t\}$ and $\{u, r\}$ are both maximal independent sets of C. Note that if w is adjacent to an additional vertex, then r must be adjacent to that vertex by claw-freedom at w. Similarly, if t is adjacent to an additional vertex, then r must be adjacent to that vertex by claw-freedom at t. Thus by Lemma 2.2, either w is adjacent to y, or r is adjacent to an additional vertex.

Suppose $w \sim y$. Then $\{wu, wy, wr\}$ is a claw at w, contradicting the fact that G is claw-free. Hence $w \nsim y$.

Suppose r is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[q, z]$ containing u, so that $V(C) = \{u, v, w, t\}$. Then C is not well-covered since $\{u\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that if $q \sim w$ then $q \sim t$ by claw-freedom at w. Thus by Lemma 2.2, either q is adjacent to z, q is adjacent to t, w is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose $q \sim z$. To prevent $\{zx, zs, zq\}$ from forming a claw at z, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing u, so that $V(C) = \{u, v, w, t, r\}$. Then C is not well-covered since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, t is adjacent to an additional vertex, or r is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv, wt, wp\}$ and $\{wv, wr, wp\}$ from forming claws at w, we must have $t \sim p$ and $r \sim p$. Thus p must be in the wrt-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus w is not adjacent to an additional vertex and so by 3-connectivity, there are no additional vertices in the wrt-face. Suppose t is adjacent to an additional vertex; call it p. To prevent $\{tu, tr, tp\}$ from forming a claw at t, we must have $r \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Recall that $s \nsim t$ by birth and so $s \nsim w$ by claw-freedom at w, and note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex. Suppose r is adjacent to an additional vertex; call it p. To prevent $\{rw, rq, rp\}$ from forming a claw at r, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p,z]$ containing u, so that $V(C) = \{u,v,w,t\}$. Then C is not well-covered since $\{u\}$ and $\{v,t\}$ are both maximal independent sets of C. Note that $p \nsim z$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $q \nsim z$.

Suppose $q \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[q,s]$ containing u, so that $V(C) = \{u,v,w,x\}$. Then C is not well-covered since $\{u\}$ and $\{w,x\}$ are both maximal independent sets of C. Note that $q \nsim w$ or we would have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus by Lemma 2.2, either q is adjacent to s, or w is adjacent to an additional vertex.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing y, so that $V(C) = \{v,x,y,z,s\}$. Then C is not well-covered since $\{y\}$ and $\{x,s\}$ are both maximal independent sets of C. Recall $s \nsim t$ by birth, and t is not adjacent to y or z by Claims 2.4/1.3.4.1.2.1.2 and 2.4/1.3.4.1.2.1.1 respectively. Thus by Lemma 2.2, either y is adjacent to an additional vertex, z is adjacent to an additional vertex, or s is adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv,yz,yp\}$ and $\{yv,ys,yp\}$ from forming claws at y, we must have $z \sim p$ and $s \sim p$. Thus p must be interior to the yzs-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus y is not adjacent to an additional vertex. Suppose z is adjacent to an additional vertex; call it p, To prevent $\{zx,zs,zp\}$ from forming a claw at z, we must have $s \sim p$. To prevent $\{sy,sp,sq\}$ from forming a claw at s, we must have $p \sim q$. (Recall that $q \nsim y$ by birth.) See Figure 37(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r,p]$ containing v, so that $V(C) = \{u,v,x,y\}$. Then C is not well-covered since $\{v\}$ and $\{u,y\}$ are both maximal independent sets of C. Note that $p \nsim r$ by planarity, and C cannot

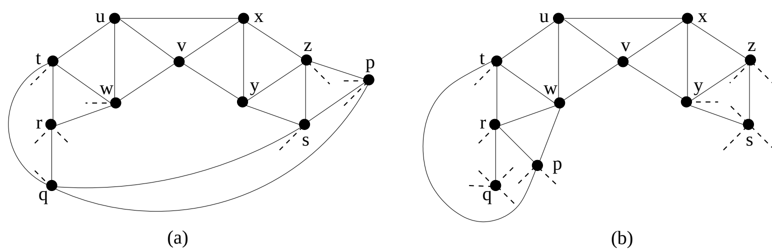


Figure 37: Proving that every vertex of degree four must lie on a K_4.

grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex. Suppose s is adjacent to an additional vertex; call it p. To prevent $\{sz, sq, sp\}$ from forming a claw at s, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[r, p]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, z\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, we must have $p \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex and so $q \nsim s$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv, wt, wp\}$ from forming a claw at w, we must have $t \sim p$. But then $\{tu, tp, tq\}$ is a claw at t, since u cannot grow and $p \nsim q$ by birth. This contradicts the fact that G is claw-free. Hence w is not adjacent to an additional vertex and so $q \nsim t$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv, wt, wp\}$ and $\{wv, wr, wp\}$ from forming claws at w, we must have $t \sim p$ and $r \sim p$. If p is in the wrt-face, then we have the forbidden subgraph shown in Figure 5(b) and centered at w. Thus p is not in the wrt-face. Suppose p is in the exterior face. See Figure 37(b) for an illustration. If p is not adjacent to a vertex interior to the trp-face, then

$\{r, t\}$ is a 2-cut, separating q from the rest of the graph and contradicting the fact that G is 3-connected. If p is not adjacent to a vertex exterior to the trp-face, then $\{t, w\}$ is a 2-cut, separating r, p and q from the rest of the graph and contradicting the fact that G is 3-connected. Thus p must have an additional neighbor interior to the trp-face and an additional neighbor exterior to the trp-face. But then these two additional neighbors together with w form a claw at p, contradicting the fact that G is claw-free. Thus w is not adjacent to an additional vertex and so by 3-connectivity, there are no additional vertices in the wrt-face.

Suppose t is adjacent to an additional vertex; call it p. To prevent $\{tp, tr, tu\}$ from forming a claw at t, we must have $p \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth, and C cannot grow. Thus by Lemma 2.2, either p is adjacent to s or q is adjacent to s. Suppose $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing u, so that $V(C) = \{u, v, w, x, t\}$. Then C is not well-covered since $\{u\}$ and $\{t, x\}$ are both maximal independent sets of C. Note that $q \nsim s$ or $\{sy, sq, sp\}$ is a claw at s, contradicting the fact that G is claw-free. (Recall $q \nsim y$ by birth.) Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it q. To prevent $\{tu, tp, tq\}$ and $\{tu, tr, tq\}$ from forming claws at t, we must have $p \sim q$ and $r \sim q$. Thus q must be in the tpr-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at t. Hence t is not adjacent to an additional vertex and so $p \nsim s$. Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall $p \nsim s$ by the preceding subcase, and note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus $q \nsim s$, and so t is not adjacent to an additional vertex, and therefore r is not adjacent to an additional vertex. Hence $s \nsim y$ as a subcase of $r \sim w$.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, yz, yq\}$ from forming a claw at y, we must have $z \sim q$. But then $\{zx, zq, zs\}$ is a claw at z, since x cannot grow and $s \nsim q$ by birth. This contradicts the fact that G is claw-free, and thus y is not adjacent to an additional vertex and so $r \nsim w$. By a similar argument, we may also assume that $s \nsim y$.

Suppose w is adjacent to an additional vertex; call it q. To prevent $\{wv, wt, wq\}$ from forming a claw at w, we must have $t \sim q$. But then $\{tu, tq, tr\}$ is a claw at t, since u cannot grow and $q \nsim r$ by birth. This contradicts the fact that G is claw-free, and thus w is not adjacent to an additional vertex (as a subcase of t adjacent to an additional vertex). By a similar argument, we may assume that y is not adjacent to an additional vertex, and therefore t is not adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it r. But then $\{wv, wr, wt\}$ is a claw at w, since v cannot grow, r is an additional vertex, and t is not adjacent to an additional vertex by the preceding subcase. This contradicts the fact that G is claw-free, and hence w is not adjacent to an additional vertex, and therefore z is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.1.2.1.3. ■

Claim 2.4/1.3.4.1.2.1.4: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.2.1.4: By way of contradiction, suppose y is adjacent to an additional vertex; call it s. See Figure 38(a) for an illustration. But then $\{yv, yz, ys\}$ is a claw at y, since v cannot grow and z is not adjacent to an additional vertex. This contradicts the fact that G is claw-free, and hence y is not adjacent to an additional vertex. ■

Therefore Claims 2.4/1.3.4.1.2.1.1, 2.4/1.3.4.1.2.1.2, 2.4/1.3.4.1.2.1.3, and 2.4/1.3.4.1.2.1.4 have eliminated all possibilities when u is adjacent to an additional

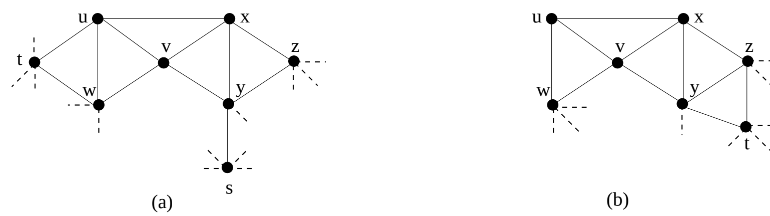


Figure 38: Proving that every vertex of degree four must lie on a K_4.

vertex. Thus u is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.1.2.1. ■

Claim 2.4/1.3.4.1.2.2: The vertex u must be adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.1.2.2: Suppose, by way of contradiction, that u is not adjacent to an additional vertex and so $d(u) = 3$. Either y is adjacent to an additional vertex, or $d(y) = 3$.

Suppose y is adjacent to an additional vertex; call it t. To prevent $\{yv, yz, yt\}$ from forming a claw at y, we must have $z \sim t$. See Figure 38(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to t, or w is adjacent to an additional vertex.

Suppose $w \sim t$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(e). Since G is not a graph from Figure 2, this graph must grow. If $w \sim y$, then we have the forbidden 5-wheel subgraph centered at y. Thus we may assume $w \nsim y$. Suppose y is adjacent to an additional vertex; call it s. Then to prevent $\{yv, yz, ys\}$ and $\{yv, yt, ys\}$ from forming claws at y, we must have $z \sim s$ and $t \sim s$. Thus s is in the zyt-face by planarity. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus y is not adjacent to an additional vertex. Hence since $w \nsim y$ and y is not adjacent to an additional vertex,

y cannot grow and $d(y) = 4$. Note that if z is adjacent to an additional vertex, then t must also be adjacent to that vertex by claw-freedom at z. Thus, since the graph must grow, either w is adjacent to z, or t is adjacent to an additional vertex.

Suppose $w \sim z$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(c). Since G is not a graph from the Figure 2, this graph must grow. Since there are no additional edges between known vertices, w, z or t must be adjacent to an additional vertex. Note that by claw-freedom at w, if w is adjacent to an additional vertex, then t must be adjacent to that vertex as well. Similarly, if z is adjacent to an additional vertex, then t must be adjacent to that vertex as well. Hence since the graph must grow, t must be adjacent to an additional vertex; call it s. To prevent $\{ty, ts, tw\}$ from forming a claw at t, we must have $s \sim w$. To prevent $\{wu, wz, ws\}$ from forming a claw at w, we must have $s \sim z$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at z. Thus $w \nsim z$.

Suppose t is adjacent to an additional vertex; call it s. To prevent $\{ty, ts, tw\}$ from forming a claw at t, we must have $s \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[s]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, u and y cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus t is not adjacent to an additional vertex, and therefore, $w \nsim t$.

Suppose w is adjacent to an additional vertex; call it s. See Figure 39(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $s \nsim t$ by birth, $s \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either w is adjacent to y, w is adjacent to an additional vertex, or s is adjacent to an additional vertex.

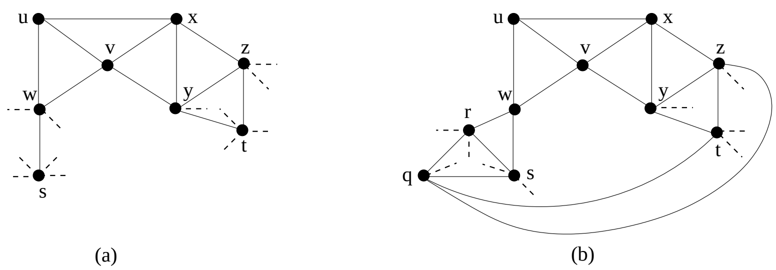


Figure 39: Proving that every vertex of degree four must lie on a K_4.

Suppose $w \sim y$. But then $\{yt, yw, yx\}$ is a claw at y since x cannot grow and $w \nsim t$ by the preceding case, which contradicts the fact that G is claw-free. Hence $w \nsim y$.

Suppose w is adjacent to an additional vertex; call it r. To prevent $\{wv, ws, wr\}$ from forming a claw at w, we must have $s \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s, r\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, r is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it q. To prevent $\{wv, ws, wq\}$ and $\{wv, wr, wq\}$ from forming claws at w, we must have $s \sim q$ and $r \sim q$. Now $d(w) = 5$, and either q is interior to the wrs-face, r is interior to the wsq-face, or s is interior to the wrq-face. By Claim 2.4/1.2, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing w, so that $V(C) = \{u, w, s, r, q\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u, and one of s, r and q, cannot grow, and u is not adjacent to any of s, r, or q. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (as a fifth neighbor), and thus there are no additional vertices in the wrs-face by 3-connectivity.

Suppose r is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $y \nsim q$ by birth, and since $s \nsim t$ by birth, $s \nsim y$ by claw-freedom at y. Also since $w \nsim t$ by a previous subcase (of the y adjacent to an additional vertex subcase, a subcase among the subcases of Claim 2.4/1.3.4.1.2.2), $w \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either q is adjacent to s, or s is adjacent to an additional vertex.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t, q]$ containing u, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{u\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, q must be adjacent to t. Since G is 3-connected, $\{w, t\}$ is not a 2-cut, and so there must be a path from y and z to the vertex set $\{q, r, s\}$ that does not pass through w or t. Note that r and q are not adjacent to y by birth, and since $s \nsim t$ by birth, $s \nsim z$ by claw-freedom at z. Thus either r is adjacent to z, q is adjacent to z, or z is adjacent to an additional vertex in the $ztqrwux$-face. Suppose $r \sim z$. To prevent $\{zx, zt, zr\}$ from forming a claw at z, we must have $r \sim t$. But then $\{rw, rq, rz\}$ is a claw at r, since w cannot grow and $q \nsim z$ by planarity. Thus $r \nsim z$. Suppose $q \sim z$. See Figure 39(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, w and x cannot grow, and $w \nsim x$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim z$. Suppose z is adjacent to an additional vertex in the $ztqrwux$-face; call it p. To prevent $\{zx, zt, zp\}$ from forming a claw at z, we must have $t \sim p$. To prevent $\{tp, ty, tq\}$ from forming a claw at t, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, s]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim s$ by planarity. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Hence n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus z is not adjacent to an additional vertex, and hence $q \nsim s$.

Suppose s is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, t]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth, and C cannot grow. Thus by Lemma 2.2, either q is adjacent to t or p is adjacent to t.

Suppose $q \sim t$. See Figure 40(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, s]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall $q \nsim s$ by the preceding subcase. Thus by Lemma 2.2, either q is adjacent to z, y is adjacent to an additional vertex, or z is adjacent to an additional vertex. Suppose $q \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then all the vertices of $C - v$ are adjacent to v, vertices v, w and x cannot grow, and $w \nsim x$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim z$. Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yz, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Thus n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus y is not adjacent to an additional vertex, and by 3-connectivity, there are no additional vertices in the yzt-face. Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zx, zt, zn\}$ from forming a claw at z, we must have $t \sim n$. But then $\{tn, ty, tq\}$ is a claw at t, since $n \nsim q$ by birth. This contradicts the fact that G is claw-free, and hence z is not adjacent to an additional vertex, so $q \nsim t$.

Suppose $p \sim t$. Note that if y is adjacent to an additional vertex; call it n, then to prevent $\{yv, yz, yn\}$ and $\{yv, yt, yn\}$ from forming claws at y, we must have $z \sim n$ and $t \sim n$. Thus n must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus y is not adjacent to an additional vertex, and by 3-connectivity, there are no additional vertices in the yzt-face. Call this semi-known subgraph S. Let C be the component of $S - N[p, r]$ containing x, so that $V(C) = \{u, v, x, y, z\}$. Then all the vertices of $C - x$ are adjacent to x, vertices x, u and y cannot grow, and $u \nsim y$. Note that since $p \nsim q$ by birth, $p \nsim r$ by claw-freedom at r. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $p \nsim t$, and so s is not adjacent to an additional vertex (as a subcase of r adjacent to an additional vertex), and therefore r is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $q \nsim y$ by birth, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex (as a subcase of w adjacent to an additional vertex), and so w is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[r, t]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, r is adjacent to t. See Figure 40(b) for an illustration. Suppose y is adjacent to an additional

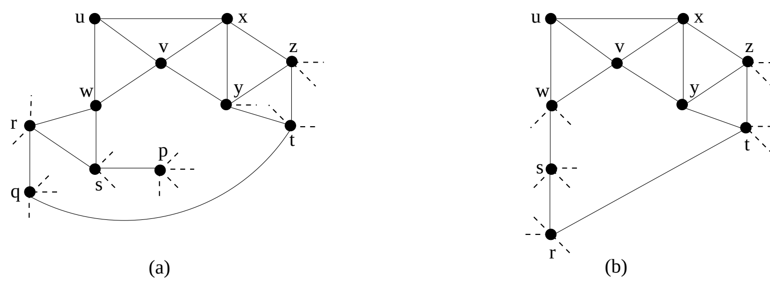


Figure 40: Proving that every vertex of degree four must lie on a K_4.

vertex; call it q. Then to prevent $\{yv, yz, yq\}$ and $\{yv, yt, yq\}$ from forming claws at y, we must have $z \sim q$ and $t \sim q$. Thus q must be in the yzt-face. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Thus y is not adjacent to an additional vertex, $d(y) = 4$, and by 3-connectivity, there are no additional vertices in the yzt-face. Since G is 3-connected, $\{w, t\}$ is not a 2-cut, and so there must be a path from z to s and r that does not pass through either w or t. Note that since $s \nsim t$ by birth, $s \nsim z$ by claw-freedom at z. Thus either r is adjacent to z, or z is adjacent to an additional vertex.

Suppose $r \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $r \nsim z$.

Suppose z is adjacent to an additional vertex; call it q. To prevent $\{zx, zt, zq\}$ from forming a claw at z, we must have $t \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, we must have $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[s, q]$ containing v, so that $V(C) = \{u, v, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 2.2, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[q]$ containing v, so that $V(C) = \{u, v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex and so s is not adjacent to an additional vertex. Therefore w is not adjacent to an additional vertex, and finally y is not adjacent to an additional vertex, which means $d(y) = 3$.

Suppose $d(y) = 3$. Since G is 3-connected and $d(w) = 2$, w must grow. The only possible additional edge between w and other known vertices is wz. Suppose w is adjacent to z. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 1(l). Since G is not a graph from Figure 1, this graph must grow. There are no possible additional edges between known vertices, and so z or w must be adjacent to an additional vertex. But then $\{w, z\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence $w \nsim z$. Then w must be adjacent to an additional vertex, but then $\{w, z\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence $d(y) \neq 3$. But this is a contradiction, since above we showed that y must have degree three. Therefore $d(u) \neq 3$, u must be adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.1.2.2. ■

Claim 2.4/1.3.4.1.2.1 says that the vertex u must not be adjacent to an additional vertex. Claim 2.4/1.3.4.1.2.2 says that the vertex u must be adjacent to an additional vertex. Since these claims contradict one another, we must have that $d(x) \neq 4$. Therefore $z \nsim y$, and we have proved Claim 2.4/1.3.4.1.2. ■

Claim 2.4/1.3.4.1.1 proves that $z \nsim u$ and Claim 2.4/1.3.4.1.2 proves that $z \nsim y$. But then $\{xu, xy, xz\}$ is a claw at x, contradicting the fact that G is claw-free. Hence x is not adjacent to an additional vertex and $d(x) = 3$. By symmetry, we may also assume that $d(u) = 3$. The graph is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of the graph, thus the graph must grow. If either w or y is adjacent to an additional vertex, then $\{w, y\}$ is a 2-cut, separating u, v, and x from the rest of the graph, and contradicting the fact that G is 3-connected. Thus there are no additional vertices. But the only possible additional edge is wy, leaving us with a 4-wheel which is not well-covered. Hence $x \nsim u$, and we have proved Claim 2.4/1.3.4.1. ■

Note that by symmetry and Claim 2.4/1.3.4.1, we may also assume that $w \nsim y$.

Claim 2.4/1.3.4.2: The vertex x is not adjacent to w.

Proof of Claim 2.4/1.3.4.2: Suppose, by way of contradiction, that $x \sim w$. See Figure 41(a) for an illustration. Since G is 3-connected, $\{w, v\}$ is not a 2-cut, and so there must be a path from x to u that does not pass through either w or v. Since $x \nsim u$ by Claim 2.4/1.3.4.1, x must be adjacent to an additional vertex in the $xvuw$-face; call it z. Also since G is 3-connected, $\{v, x\}$ is not a 2-cut and so there must exist a path from w to y that does not pass through v or x. Since $w \nsim y$ by Claim 2.4/1.3.4.1, w must be adjacent to an additional vertex in the $xyvw$-face; call it t. If x has no additional neighbors in the $xyvw$-face, then $\{w, y\}$ is a 2-cut, separating t from the rest of the graph, and contradicting the fact that G is 3-connected. Hence x must be adjacent to a vertex in the $xyvw$-face. But then this vertex along with v and z is a claw at x contradicting the fact that G is claw-free. Hence $x \nsim w$. ■

Note that by symmetry and Claim 2.4/1.3.4.2, we may also assume that $u \nsim y$. Thus by Claim 2.4/1.3.4.1 and Claim 2.4/1.3.4.2, if a vertex has degree four and does not lie on a K_4, then its neighborhood can be divided into two disjoint sets N_1 and N_2, such that N_i is a K_2 (for $i = 1, 2$) and there are no edges between N_1 and N_2. The following claim will be helpful in our proof.

Figure 41: Proving that every vertex of degree four must lie on a K_4.

Claim 2.4/1.3.4.3: If any two adjacent vertices of G share more than one neighbor, they must be in a K_4 together with those neighbors.

Proof of Claim 2.4/1.3.4.3: Suppose that vertices u and v of G are adjacent and that they share two neighbors w and x, but w is not adjacent to x (i.e. u, v, w, and x do not span a K_4).

We claim $d(u) = 3 = d(v)$.

Suppose by way of contradiction that u is adjacent to a fourth vertex y. By claw-freedom, either y is adjacent to w or y is adjacent to x. Without loss of generality, assume that y is adjacent to w. But then we have a forbidden subgraph at u with w adjacent to v by Claim 2.4/1.3.4.1. Hence the degree of u must be three and by symmetry, the degree of v must be three also. ■

But if $d(u) = 3 = d(v)$, then either $\{w, x\}$ is a 2-cut, which is a contradiction, or $G = K_4 - e$ which is not well-dominated.

Hence u and v are in a K_4 together with w and x. ■

Claim 2.4/1.3.4.4: The vertex x is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4: Suppose by way of contradiction, that x is adjacent to an additional vertex; call it z. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either z is adjacent to y, z is adjacent to u, z is adjacent to w, y is adjacent to an additional vertex, u is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.1: The vertex z is not adjacent to y.

Proof of Claim 2.4/1.3.4.4.1: Suppose by way of contradiction, that $z \sim y$. See Figure 41(b) for an illustration. Either x is adjacent to an additional vertex, or $d(x) = 3$. Suppose x is adjacent to an additional vertex; call it t. To prevent $\{xv, xz, xt\}$ from forming a claw at x, we must have $z \sim t$. If $d(x) = 4$, then we have a forbidden subgraph centered at x, by Claim 2.4/1.3.4.1. Thus $d(x) = 5$, and x must be adjacent to an additional vertex; call it s. To prevent $\{xv, xz, xs\}$ and $\{xv, xt, xs\}$ from forming claws at x, we must have $z \sim s$ and $t \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at x. Therefore, x is not adjacent to an additional vertex and $d(x) = 3$. Since G is 3-connected, $\{z, v\}$ is not a 2-cut, and so there must be a path from y to u and w that does not pass through either z or v. Thus y must be adjacent to an additional vertex; call it t. To prevent $\{yv, yz, yt\}$ from forming a claw at y, we must have $z \sim t$. If $d(y) = 4$, then we have a forbidden subgraph centered at y, by Claim 2.4/1.3.4.1. Thus $d(y) = 5$, and y must be adjacent to an additional vertex; call it s. To prevent $\{yv, yz, ys\}$ and $\{yv, yt, ys\}$ from forming claws at y, we must have $z \sim s$ and $t \sim s$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at y. Hence the degree of x must be greater than three, a contradiction. Therefore $z \nsim y$. ■

Claim 2.4/1.3.4.4.2: The vertex z is not adjacent to u.

Proof of Claim 2.4/1.3.4.4.2: Suppose by way of contradiction, that $z \sim u$. Either $d(x) = 5$, $d(x) = 4$, or $d(x) = 3$.

Claim 2.4/1.3.4.4.2.1: The vertex x has degree less than five.

Proof of Claim 2.4/1.3.4.4.2.1: Suppose, by way of contradiction, that $d(x) = 5$. Then x must be adjacent to two additional vertices, call them t and s. To prevent $\{xv, xz, xs\}$, $\{xv, xz, xt\}$ and $\{xv, xt, xs\}$ from forming claws at x, we must have $z \sim s$, $z \sim t$, and $t \sim s$. Either t is interior to the xzs-face, or s is interior to the xzt-face. Without loss of generality, suppose s is interior to the xzt-face. Now either t and s are both interior to the $uvxz$-face, or they are both interior to the $xzuwvy$-face. Suppose t and s are both interior to the $uvxz$-face. Since G is 3-connected, $\{u, x\}$ is not a 2-cut, and so there must be a path from z to w and y that does not pass through u or x. By Claim 2.4/1.3.4.4.1, we known $z \nsim y$. Thus either z is adjacent to w, or z is adjacent to an additional vertex in the $xzuwvy$-face. Suppose $z \sim w$. See Figure 42(a) for an illustration. Since G is 3-connected, $\{x, z\}$ is not a 2-cut, and so there must be a path from t to u that does not pass through either x or z. If $u \sim t$, then we have the forbidden subgraph shown in Figure 5(b) and centered at z. Thus u must be adjacent to an additional vertex in the $uvxtz$-face. But then $\{u, t\}$ is a 2-cut, separating this additional vertex from the rest of the graph, and contradicting the fact that G is 3-connected. Thus $z \nsim w$. Suppose z is adjacent to an additional vertex; call it r. To prevent $\{zx, zu, zr\}$ from forming a claw at z, we must have $u \sim r$. Since G is 3-connected, $\{x, z\}$ is not a 2-cut, and so there must exist a path from t to u that does not pass through either x or z. If $u \sim t$, then $\{uv, ur, ut\}$ is a claw at u. Thus u must be adjacent to an additional vertex in the $uvxtz$-face. But then $\{u, t\}$ is a 2-cut, separating this additional vertex from the rest of the graph, and contradicting the fact that G is 3-connected. Hence z is not adjacent to an additional vertex, and t and s are not interior to the $uvxz$-face.

Suppose t and s are interior to the $xzuwvy$-face. Then since G is 3-connected, there are no additional vertices in the $uvxz$-face. Call this semi-known subgraph

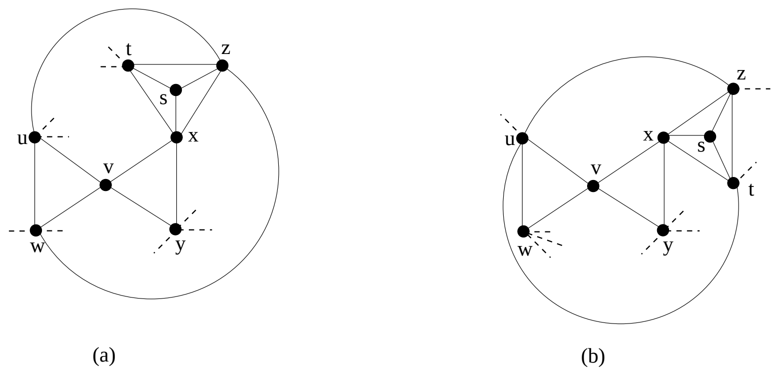


Figure 42: Proving that every vertex of degree four must lie on a K_4.

S. Let C be the component of $S - N[u]$ containing x, so that $V(C) = \{x, y, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{y, s\}$ are both maximal independent sets of C. Recall that $u \nsim y$ by Claim 2.4/1.3.4.2, and $u \nsim s$ by planarity. Also note that $y \nsim t$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at x. Thus by Lemma 2.2, either u is adjacent to t, vertex t is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.2.1.1: The vertex u is not adjacent to t.

Proof of Claim 2.4/1.3.4.4.2.1.1: Suppose, by way of contradiction, that $u \sim t$. See Figure 42(b) for an illustration. Suppose u is adjacent to an additional vertex in the $uwvyxt$-face; call it r. Then $\{uv, uz, ur\}$ is a claw at u, contradicting the fact that G is claw-free. Hence u is not adjacent to an additional vertex in the $uwvyxt$-face. Recall that $u \nsim y$ by Claim 2.4/1.3.4.2. Note that $t \nsim w$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at t. Also note that $y \nsim t$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at x. Suppose t is adjacent to an additional vertex in the $uwvyxt$-face; call it r. Then $\{tx, tu, tr\}$ is a claw at t, contradicting the fact that G is claw-free. Hence t is not adjacent to an additional vertex in the $uwvyxt$-face. Since G is 3-connected, $\{v, x\}$ is not a 2-cut, and so there must exist a path from y to w that does not pass through either v or x. Recall that $w \nsim y$ by Claim 2.4/1.3.4.1. Thus w must be adjacent to an additional vertex. But then $\{w, y\}$ is a 2-cut, separating this additional vertex from the rest of the graph. Therefore $u \nsim t$. ■

Claim 2.4/1.3.4.4.2.1.2: The vertex t is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.1.2: Suppose, by way of contradiction, that t is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[r, u]$ containing x, so that $V(C) = \{s, x, y\}$. Then C is not well-covered, since $\{x\}$ and $\{s, y\}$ are both maximal independent sets of C. Note that $r \nsim u$ by birth. Thus either r is adjacent to y, or y is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.2.1.2.1: The vertex r is not adjacent y.

Proof of Claim 2.4/1.3.4.4.2.1.2.1: Suppose, by way of contradiction, that $r \sim y$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut, therefore there must be a path from w to the vertex set $\{z, y, t, r\}$ that does not pass through either u or v. Note that if $w \sim r$, then $w \sim t$ otherwise $\{ry, rt, rw\}$ would be a claw at r, since $y \nsim t$ (forbidden subgraph shown in Figure 5(b) and centered at x) and $w \nsim y$ (Claim 2.4/1.3.4.1). Similarly, if $w \sim t$, then $w \sim r$. Hence either w is adjacent to z, w is adjacent to both r and t, or w is adjacent to an additional vertex.

Suppose $w \sim z$. See Figure 43(a) for an illustration. Note that $d(z) = 5$. Then $w \nsim t$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at z. Thus $w \nsim r$; otherwise $\{ry, rt, rw\}$ would be a claw at r. Suppose w is adjacent to an additional vertex in the $wvyrtz$-face; call it q. Then $\{wv, wz, wq\}$ is a claw at w, contradicting the fact that G is claw-free. So w is not adjacent to an additional vertex in the $wvyrtz$-face. But then $\{y, t\}$ is a 2-cut, separating r from the rest of the graph, and contradicting the fact that G is 3-connected. Hence $w \nsim z$.

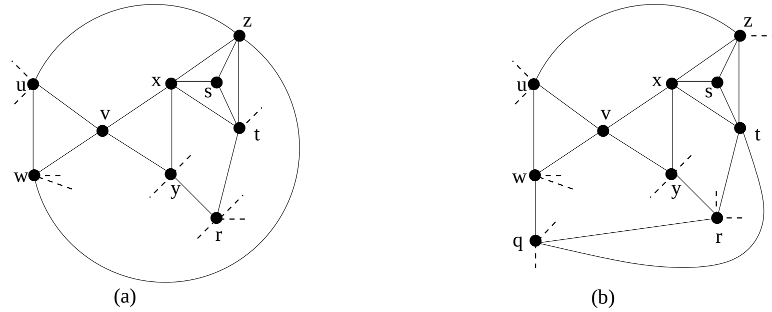


Figure 43: Proving that every vertex of degree four must lie on a K_4.

Suppose w is adjacent to both r and t. Suppose w is adjacent to an additional vertex in the $wvyr$-face; call it q. Then $\{wv, wt, wq\}$ is a claw at w, contradicting the fact that G is claw-free. Hence w is not adjacent to a vertex in the $wvyr$-face, and therefore there are no additional vertices in the $wvyr$-face by 3-connectivity. Note that $d(t) = 5$. Thus by 3-connectivity, there are no additional vertices in the $xtry$-face. Hence since $y \nsim w$ by Claim 2.4/1.3.4.1, y cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[w]$ containing x, so that $V(C) = \{x, y, z, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, y and s cannot grow, and $y \nsim s$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Therefore w is adjacent to neither r nor t.

Suppose w is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[q, t]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to t, q is adjacent to y, q is adjacent to u, y is adjacent to an additional vertex, or u is adjacent to an additional vertex.

Suppose $q \sim t$. To prevent $\{tx, tr, tq\}$ from forming a claw at t, we must have $q \sim r$. See Figure 43(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, y]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note

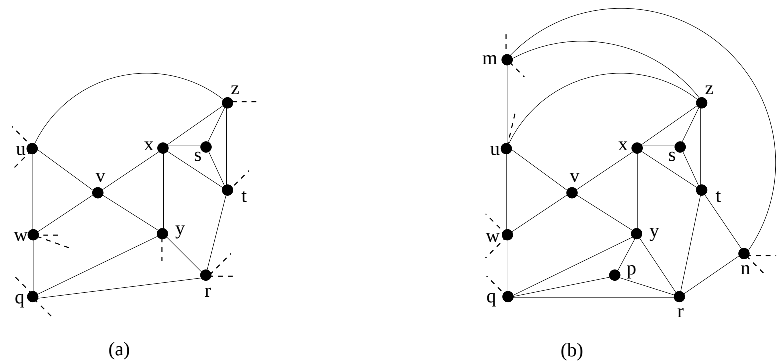


Figure 44: Proving that every vertex of degree four must lie on a K_4.

that by claw-freedom at each of z and u, if one is adjacent to another vertex, the other must also be adjacent to that vertex. Thus by Lemma 2.2, either q is adjacent to y, q is adjacent to both u and z, or u and z share an additional neighbor. Suppose $q \sim y$. Then $\{qw, qy, qt\}$ is a claw at q, contradicting the fact that G is claw-free. Thus $q \nsim y$. Suppose q is adjacent to both u and z. Then we have the forbidden subgraph shown in Figure 5(b) and centered at z. Thus q is adjacent to neither u nor z. Suppose u and z share an additional neighbor, and call it p. Call this semi-known subgraph S. Let C be the component of $S - N[q, p]$ containing x, so that $V(C) = \{v, x, y, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v, and s cannot grow, and $v \nsim s$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus neither u nor z is adjacent to an additional vertex, and therefore, $q \nsim t$.

Suppose $q \sim y$. To prevent $\{yv, yq, yr\}$ from forming a claw at y, we must have $q \sim r$. See Figure 44(a) for an illustration. Either y is adjacent to an additional vertex, or $d(y) = 4$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yr, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $r \sim p$ and $q \sim p$. Thus p must be in the yrq-face. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(h). Since G is not a graph from the Figure 2, this graph must grow. Note that $d(y) = 5$ and so, by Claim 2.4/1.2, p cannot grow. By claw-freedom, either r and t share an additional neighbor, or $d(r) = d(t) = 4$.

Suppose r and q share an additional neighbor and call it n. Since G is 3-connected, n must be in the $uwqrtz$-face. Also since G is 3-connected, $\{r, t\}$ is not a 2-cut, and so there must be a path from n to the vertex set $\{u, z, w, q\}$ that does not pass through either r or t. Note that by claw-freedom, u and z must share any other neighbors that either of them have, and similarly for w and q. Also note that $n \nsim z$ (and hence $n \nsim u$); otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at t. Also, $n \nsim q$ (and hence $n \nsim w$); otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at r. Thus either u and z share an additional neighbor, or w and q share an additional neighbor.

Suppose u and z share an additional neighbor and call it m. Call this semi-known subgraph S. Let C be the component of $S - N[n, m, q]$ containing x, so that $V(C) = \{s, x, v\}$. Note that $n \nsim q$; otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at r. Suppose $m \sim q$. To prevent $\{qp, qw, qm\}$ from forming a claw at q, we must have $w \sim m$. But then we have the subgraph centered at w and forbidden by Claim 2.4/1.3.4.1. Therefore $m \nsim q$. Note that C cannot grow. Thus by Lemma 2.2, $m \sim n$. See Figure 44(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[m, q]$ containing x, so that $V(C) = \{t, s, x, v\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus neither u nor z is adjacent to an additional vertex.

Suppose w and q share an additional neighbor; call it m. Since G is 3-connected, $\{t, r\}$ is not a 2-cut, and so there must be a path from n to m that does not pass through either t or r. Note that $d(q) = 5$, and so w is not adjacent to an additional vertex; otherwise this additional vertex along with q and v would be a claw at w, contradicting the fact that G is claw-free. Thus m and n are the only vertices that can grow. If m were adjacent to an additional vertex, then $\{m, n\}$ would be a 2-cut, separating this additional vertex from the rest of the graph, and contradicting the fact that G is 3-connected. Thus we must have $m \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[m, z]$ containing y, so that $V(C) = \{v, y, p, r\}$. Then C is not well-covered since $\{y\}$ and $\{v, p\}$ are both maximal independent sets. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus neither w nor q is adjacent to an additional vertex and so r and t do not share an additional vertex.

Suppose $d(r) = d(t) = 4$. Recall this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(h), and so since this graph must grow and by claw-freedom either u and z share an additional neighbor, or w and q share an additional neighbor. (Note that $u \nsim q$; otherwise $\{uz, uv, uq\}$ would be a claw at q, contradicting the fact that G is claw-free.)

Suppose u and z share an additional neighbor; call it n. Since G is 3-connected, $\{u, z\}$ is not a 2-cut, and so there exists a path from n to w and q that does not pass through either u or z. Thus by claw-freedom, either w and q are both adjacent to n, or w and q share an additional neighbor. Suppose w and q are both adjacent to n. Call this semi-known subgraph S. Let C be the component of $S - N[n, s]$ containing y, so that $V(C) = \{v, y, p, r\}$. Then C is not well-covered since $\{y\}$ and $\{v, p\}$ are both maximal independent sets. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus neither w nor q is adjacent to n. Suppose w and q share an additional neighbor and call it m. See Figure 45(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[n, q]$ containing x, so that $V(C) = \{v, x, s, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are

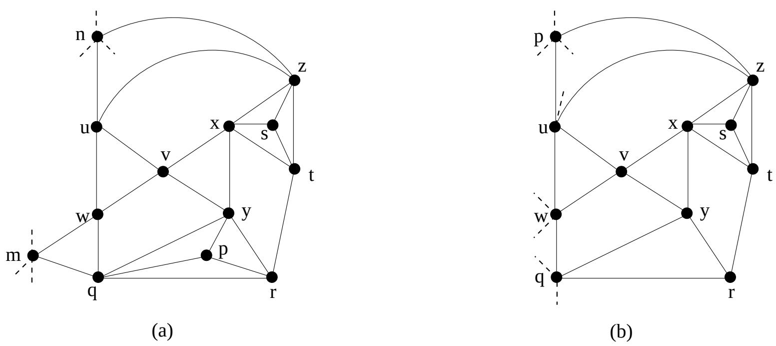


Figure 45: Proving that every vertex of degree four must lie on a K_4.

both maximal independent sets. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus neither w nor q is adjacent to an additional vertex (as a subcase to u and z sharing an additional neighbor), and so neither u nor z is adjacent to an additional vertex.

Suppose w and q share an additional neighbor, and call it n. But then $\{w, q\}$ is a 2-cut, contradicting the fact that G is 3-connected. Hence neither w nor q is adjacent to an additional vertex and so r and t must have degree greater than four. This is a contradiction, and so y is not adjacent to an additional vertex.

Suppose $d(y) = 4$. Then this semi-known subgraph is isomorphic to the exceptional well-covered graph shown in Figure 2(i). Since G is not a graph from the Figure 2, this semi-known subgraph must grow. Note that if $z \sim q$ then $\{qw, qy, qz\}$ is a claw at q, thus $z \nsim q$. Note that if $q \sim t$ then $\{qy, qw, qt\}$ is a claw at q, thus $q \nsim t$. Thus there are no possible additional edges between known vertices, and so there must be additional vertices. Thus by claw-freedom either r and t share an additional neighbor, u and z share an additional neighbor, or q and w share an additional neighbor.

Suppose r and t share an additional neighbor; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p, w]$ containing x, so that $V(C) =$

$\{x, y, z, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, y and s cannot grow, and $y \nsim s$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence neither r nor t is adjacent to an additional vertex.

Suppose u and z share an additional neighbor; call it p. See Figure 45(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p, q]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Hence neither u nor z is adjacent to an additional vertex.

Suppose q and w share an additional neighbor; call it p. But then $\{q, w\}$ is a 2-cut, separating p from the rest of the graph, and contradicting the fact that G is 3-connected. Hence neither q nor w is adjacent to an additional vertex. Therefore $q \nsim y$.

Suppose $q \sim u$. To prevent $\{uv, uz, uq\}$ from forming a claw at u, we must have $q \sim z$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim u$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yr, yp\}$ from forming a claw at y, we must have $p \sim r$. See Figure 46(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, y]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered, since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $q \nsim u$ by the preceding subcase, $q \nsim z$ by claw-freedom at z. Recall that q is not adjacent to either y or t by previous subcases (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.1). Thus by Lemma 2.2, either u and z share an additional neighbor, or t is adjacent to an additional vertex. Suppose u and z share an additional

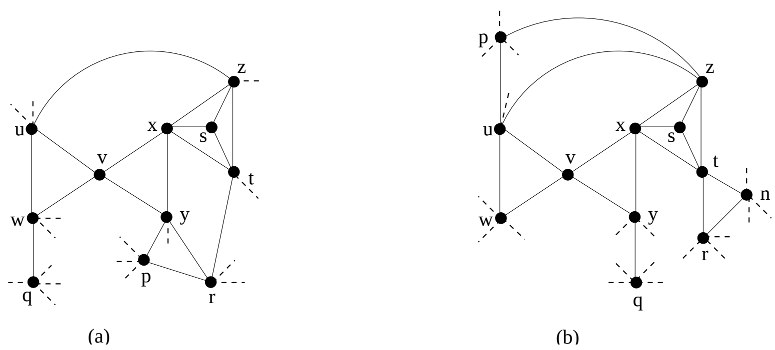


Figure 46: Proving that every vertex of degree four must lie on a K_4.

neighbor, and call it n. Call this semi-known subgraph S. Let C be the component of $S - N[n, r, q]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $n \nsim q$ by birth, and $n \nsim r$; otherwise $\{ry, rt, rn\}$ is a claw at r, contradicting the fact that G is claw-free. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither u nor z is adjacent to an additional vertex. Suppose t is adjacent to an additional vertex; call it n. To prevent $\{tx, tr, tn\}$ from forming a claw at t, we must have $n \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[n, y, q]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered, since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that n is not adjacent to either q or y by birth, and C cannot grow by the preceding subcase. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence, t is not adjacent to an additional vertex, and therefore y is not adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it p. To prevent $\{uv, uz, up\}$ from forming a claw at u, we must have $z \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[r, p, q]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $p \nsim q$ by birth. Since $p \nsim t$ by birth and y is not adjacent to an additional vertex (which includes p) by the preceding case, $p \nsim r$ by claw-freedom at r. Also note that since q is not adjacent to t or y by previous subcases (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.1), $q \nsim r$ by claw-freedom at r. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence u is not adjacent to an additional vertex, and therefore w is not adjacent to an additional vertex. Thus $r \nsim y$. ■

Claim 2.4/1.3.4.4.2.1.2.2: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.1.2.2: Suppose, by way of contradiction that y is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[q, v]$ containing t, so that $V(C) = \{z, t, s, r\}$. Then C is not well-covered since $\{t\}$ and $\{r, s\}$ are both maximal independent sets of C. Note that $q \nsim r$ by birth. Thus by Lemma 2.2, either z is adjacent to an additional vertex, t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zx, zu, zp\}$ from forming a claw at z, we must have $u \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[u, y]$ containing t, so that $V(C) = \{t, s, r\}$. Then C is not well-covered since $\{t\}$ and $\{r, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it n. To prevent $\{tx, tr, tn\}$ from forming a claw at t we must have $r \sim n$. See Figure 46(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, y]$ containing t, so that $V(C) = \{t, s, r, n\}$. Then C is not well-covered since $\{t\}$ and $\{r, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it m. Note that $m \nsim n$ by Claim

2.4/1.3.4.3, since t cannot grow, and so r and n cannot share another neighbor. Call this semi-known subgraph S. Let C be the component of $S - N[u, y, m]$ containing t, so that $V(C) = \{t, s, n\}$. Then C is not well-covered since $\{t\}$ and $\{n, s\}$ are both maximal independent sets of C. Note that m is not adjacent to y or u by birth. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[m, k, y, w]$ containing z, so that $V(C) = \{p, z, s, t\}$. Then C is not well-covered, since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Note that k is not adjacent to either m or y by birth, and $w \nsim p$; otherwise we would have a forbidden subgraph by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either k is adjacent to w, m is adjacent to w, k is adjacent to p, m is adjacent to p, or p is adjacent to an additional vertex.

Suppose $k \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[m, k, y]$ containing z, so that $V(C) = \{u, p, z, s, t\}$. Then every vertex of $C - z$ is adjacent to z, vertices z, u, and s cannot grow, and further $u \nsim s$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $k \nsim w$.

Suppose $m \sim w$. See Figure 47(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[m, k, y]$ containing z, so that $V(C) = \{u, p, z, s, t\}$. Then every vertex of $C - z$ is adjacent to z, vertices z, u, and s cannot grow, and $u \nsim s$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $m \nsim w$.

Suppose $p \sim k$. Since G is 3-connected, $\{u, v\}$ is not at 2-cut, and so there must be a path from w to the vertex set $\{y, r, q, p, n, m, k\}$ that does not pass through either u or v. Since w is not adjacent to either m or k by the two preceding subcases, w is not adjacent to n or r by claw-freedom at each respectively. Thus either w is adjacent to q or w is adjacent to an additional vertex.

Suppose $q \sim w$. Call this semi-known subgraph S. Let C be the component of

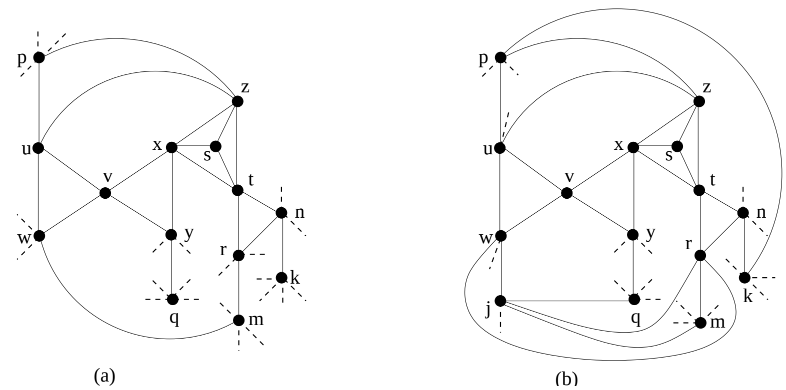


Figure 47: Proving that every vertex of degree four must lie on a K_4.

$S - N[q,p,r]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Recall that $p \nsim q$ by birth and $q \nsim r$ by birth. Since $r \nsim k$ by Claim 2.4/1.3.4.3, $r \nsim p$ by claw-freedom at p. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim q$.

Suppose w is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[q,p,r,j]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either j is adjacent to q, j is adjacent to p, or j is adjacent to r. Suppose $j \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p,r,j]$ containing x, so that $V(C) = \{v,x,y,s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and s cannot grow, and $v \nsim s$. Thus if $\{p,r,j\}$ is an independent set of vertices, by Lemma 2.3, G is not well-covered, a contradiction. Thus $\{p,r,j\}$ is not independent. Since p is not adjacent to either w or q, $p \nsim j$ by claw-freedom at j. Thus we must have $j \sim r$. To prevent $\{rt, rm, rj\}$ from forming a claw at r, we must have $j \sim m$. To prevent $\{jw, jq, jr\}$ from forming a claw at j, we must have $r \sim w$ (recall $q \nsim r$ by birth). See Figure 47(b) to see a picture. But then $\{rt, rw, rm\}$ is a claw at r, since $m \nsim w$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2). This contradicts the fact that G is claw-free, and hence $j \nsim q$. Suppose $j \sim p$. To prevent $\{pj, pu, pk\}$ from forming a claw at p, we must have $k \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $j \sim r$. To prevent $\{jr, jw, jp\}$ from forming a claw at j, we must have $w \sim r$. But then $\{rt, rw, rm\}$ is a claw at r, contradicting the fact that G is claw-free, and hence $j \nsim p$. Suppose $j \sim r$. To prevent $\{rt, rm, rj\}$ from forming a claw at r, we must have $j \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, k, j]$ containing z, so that $V(C) = \{u, z, s, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Since $j \nsim q$ by on of the two preceding subcases, $j \nsim y$ by claw-freedom at y. Since $m \nsim k$ by birth, and w is not adjacent to k or m by previous subcases (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $j \nsim k$ otherwise $\{jw, jm, jk\}$ would be a claw at j, contradicting the fact that G is claw-free. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $j \nsim r$, and so w is not adjacent to an additional vertex, and therefore $p \nsim k$.

Suppose $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[w, y, r, k]$ so that $V(C) = \{p, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Recall that $m \nsim y$ by birth, $k \nsim m$ by birth, and $m \nsim w$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2). Since w, y and k are not adjacent to m, none of them are adjacent to r or p by claw-freedom at r or p

respectively. Thus by Lemma 2.2, either p is adjacent to r, or p is adjacent to an additional vertex.

Suppose $p \sim r$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut, and so there must be a path from w to the vertex set $\{y, r, q, p, n, m, k\}$ that does not pass through u or v. Recall that w is not adjacent to k or m by previous subcases (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2). Thus w is not adjacent to r, p or n by claw-freedom at those vertices. Hence either w is adjacent to q, or w is adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, n]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since $p \nsim k$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $p \nsim n$ by claw-freedom at n. Thus by Lemma 2.2, q must be adjacent to n. To prevent $\{nt, nk, nq\}$ from forming a claw at n, we must have $q \sim k$. See Figure 48(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, p]$ containing x, so that $V(C) = \{v, x, s, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim q$.

Suppose w is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, n, j]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either j is adjacent to q, j is adjacent to p, j is adjacent to n, or n is adjacent to q. Suppose $j \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, n, j]$ containing x, so that $V(C) = \{v, x, y, s\}$. Note that since p is not adjacent to q or w, $p \nsim j$ by claw-freedom at j. Then every vertex of $C - x$ is adjacent to x, vertices x, v, and s cannot grow, and $v \nsim s$. Thus if $\{p, n, j\}$ is an independent set, then by Lemma 2.3, G is not well-covered, a contradiction. Hence $\{p, n, j\}$ must be dependent and so $j \sim n$. To prevent $\{jw, jq, jn\}$ form forming a claw at j, we must have $n \sim q$. To prevent $\{nj, nt, nk\}$ from forming a claw at n, we must have $j \sim k$. To prevent $\{nq, nt, nk\}$ from forming a claw at n, we must have $k \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, j]$ containing x, so that $V(C) = \{v, x, y, s, t\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v, and s cannot grow, and $v \nsim s$. Thus by Lemma 2.3 G is not well-covered, a contradiction. Hence $j \nsim q$. Suppose $j \sim p$. To prevent $\{pu, pr, pj\}$ and $\{pu, pm, pj\}$ from forming claws at p, we must have $r \sim j$ and $m \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $j \nsim q$ by the preceding subcase, $j \nsim y$ by claw-freedom at y. Also $j \nsim n$ otherwise $\{jw, jp, jn\}$ would be a claw at j. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $j \nsim p$. Suppose $j \sim n$. To prevent $\{nt, nj, nk\}$ from forming a claw at n, we must have $j \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, j]$ containing x, so that $V(C) = \{v, x, s, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $j \nsim n$. Suppose $n \sim q$. To prevent $\{nt, nq, nk\}$ from forming a claw at n, we must have $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, j]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence

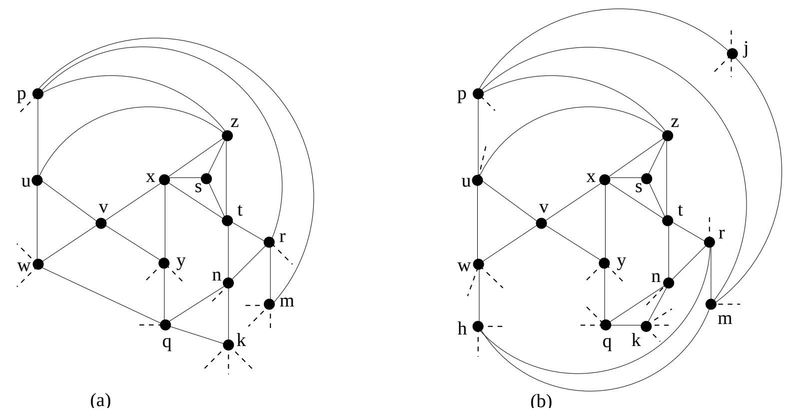


Figure 48: Proving that every vertex of degree four must lie on a K_4.

$n \nsim q$, and so w is not adjacent to an additional vertex, and therefore $p \nsim r$.

Suppose p is adjacent to an additional vertex; call it j. To prevent $\{pu, pm, pj\}$ from forming a claw at p, we must have $m \sim j$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must exist at path from w to the vertex set $\{y, r, q, p, n, m, k, j\}$ that does not pass through u or v. Note that $j \nsim w$ by birth. Thus either w is adjacent to q, or w is adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, r, k]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since $k \nsim y$ by birth and $k \nsim w$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $k \nsim q$ by claw-freedom at q. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim w$.

Suppose w is adjacent to an additional vertex; call it h. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, n, h]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either h is adjacent to q, h is adjacent to p, h is adjacent to n, or n is adjacent to q.

Suppose $h \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, n, h]$ containing x, so that $V(C) = \{v, x, y, s\}$. Note that since p is not adjacent to q or w, $p \nsim h$ by claw-freedom at h. Then every vertex of $C-x$ is adjacent to x, vertices x, v, and s cannot grow, and $v \nsim s$. Thus if $\{p, n, h\}$ is an independent set, then by Lemma 2.3, G is not well-covered, a contradiction. Hence $\{p, n, h\}$ must be a dependent set and so $h \sim n$. To prevent $\{hw, hq, hn\}$ from forming a claw at h, we must have $n \sim q$. To prevent $\{nh, nt, nk\}$ from forming a claw at n, we must have $h \sim k$. To prevent $\{nq, nt, nk\}$ from forming a claw at n, we must have $k \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, r, h]$ containing x, so that $V(C) = \{v, x, y, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v, and s cannot grow, and $v \nsim s$. Note that since r is not adjacent to w or q, $r \nsim h$ by claw-freedom at h. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $h \nsim q$.

Suppose $h \sim p$. To prevent $\{pu, pj, ph\}$ and $\{pu, pm, ph\}$ from forming claws at p, we must have $j \sim h$ and $m \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, h]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $h \nsim q$ by the preceding subcase, $h \nsim y$ by claw-freedom at y. Also $h \nsim n$; otherwise $\{hw, hp, hn\}$ would be a claw at h. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $h \nsim p$.

Suppose $h \sim n$. To prevent $\{nt, nh, nk\}$ from forming a claw at n, we must have $h \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[p, q, r, h]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since $h \sim n$, by Claim 2.4/1.3.4.3 $h \nsim r$. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $h \nsim n$.

Suppose $n \sim q$. To prevent $\{nt, nq, nk\}$ from forming a claw at n, we must have $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, r, h]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, $h \sim r$. To prevent $\{rt, rm, rh\}$ from forming a claw at r, we must have $h \sim m$. See Figure 48(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, h]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim q$, and so w is not adjacent to an additional vertex, and therefore p is not adjacent to an additional vertex (as a subcase of the $m \sim p$ case). Thus $m \nsim p$.

Suppose p is adjacent to an additional vertex; call it j. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must exist a path from w to the vertex set $\{y, r, q, p, n, m, k, j\}$ that does not pass through u or v. Note that $j \nsim w$ by birth. Thus either w is adjacent to q or w is adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, m, k]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since m and k are not adjacent to w or y, neither m nor k is adjacent to q by claw-freedom at q. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim q$.

Suppose w is adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, r, i]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since $m \nsim p$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $p \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either i is adjacent to p, i is adjacent to q, or i is adjacent to r.

Suppose $i \sim p$. To prevent $\{pi, pj, pu\}$ from forming a claw at p, we must have $i \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[i, y, m, k]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $j \nsim y$ by birth, $y \nsim i$ by claw-freedom at i. Also note that since j is not adjacent to m or k by birth, i is not adjacent to m or k by claw-freedom at i. Finally, note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $i \nsim p$.

Suppose $i \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[i, y, r, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $i \nsim r$; otherwise by claw-freedom at i we would have $r \sim w$, but then by claw-freedom at r, we would need $m \sim w$, but this is a contradiction since $m \nsim w$ by a previous subcase (of the r adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2). Since $j \nsim m$ by birth, $j \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either i is adjacent to y or i is adjacent to j. Suppose $i \sim y$. See Figure 49(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[i, p, r]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $i \nsim y$. Suppose $i \sim j$. To prevent $\{iw, iq, ij\}$ from forming a claw at i, we must have $j \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[i, p, r, k]$

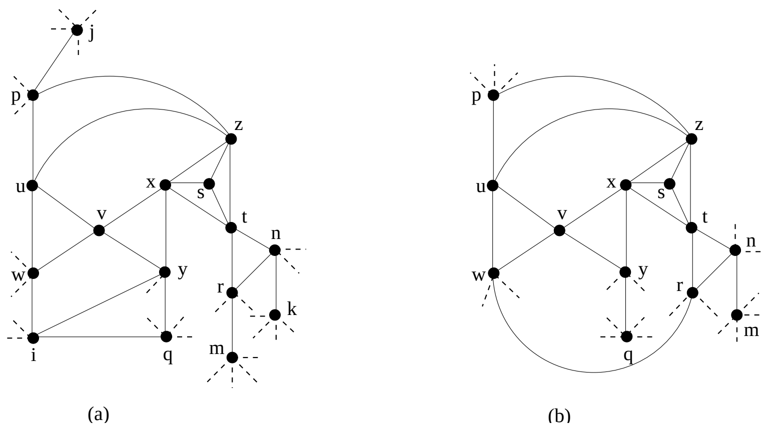


Figure 49: Proving that every vertex of degree four must lie on a K_4.

containing x, so that $V(C) = \{v, x, y, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, s and v cannot grow, and $v \nsim s$. Note that $i \nsim k$ by planarity. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $i \nsim j$ and so $i \nsim q$.

Suppose $i \sim r$. To prevent $\{ri, rt, rm\}$ from forming a claw at r, we must have $i \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[i, p, q, n]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Since k is not adjacent to m or w, $k \nsim i$ by claw-freedom at i. Hence $i \nsim n$ by claw-freedom at n. Thus by Lemma 2.2, we must have $n \sim q$. To prevent $\{nt, nk, nq\}$ from forming a claw at n, we must have $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[i, p, q]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered, since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $i \nsim r$, and so w is not adjacent to an additional vertex and therefore p is not adjacent to an additional vertex. Thus r is not adjacent to an additional vertex.

Suppose n is adjacent to an additional vertex; call it m. Since $\{n, t\}$ is not a 2-cut, there must be a path from r to the vertex set $\{w, y, q, p, m\}$. Since r is not adjacent to an additional vertex by the preceding subcase and m is an additional vertex, $m \nsim r$. Recall that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1, and $r \nsim q$ by birth. Hence either r is adjacent to w or r is adjacent to p. Suppose $r \sim w$. See Figure 49(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r, p, q]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $p \nsim r$; otherwise $\{pr, pw, pt\}$ would be a claw at p, since $p \nsim w$ by Claim 2.4/1.3.4.1. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $r \nsim w$. Suppose $r \sim p$. Since $r \nsim w$ and r is not adjacent to an additional vertex, $d(r) = 3$. Suppose $p \sim n$. Then to prevent $\{nt, nm, np\}$ from forming a claw at n, we must have $p \sim m$. But then $\{pu, pm, pr\}$ is a claw at p, contradicting the fact that G is claw-free. Hence $p \nsim n$. Similarly, p is not adjacent to an additional vertex; otherwise the additional neighbor along with r and u would form a claw at p. Hence $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[w, y, n]$ containing z, so that $V(C) = \{z, s, p\}$. Then C is not well-covered, since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $w \sim n$. To prevent $\{nt, nm, nw\}$ from forming a claw at n, we must have $w \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing z, so that $V(C) = \{u, z, s, p\}$. Then C is not well-covered, since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $r \nsim p$, and so n is not adjacent to an additional vertex. Therefore t is not adjacent to an additional vertex.

Note that $t \nsim p$; otherwise we would have the forbidden subgraph as shown in Figure 5(b) and centered at z. Since y and q are not adjacent to r, neither y nor q is adjacent to t. Therefore, with the above result we have that either t is adjacent to w, or $d(t) = 4$.

Suppose r is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[n, y, w]$ containing z, so that $V(C) = \{z, t, s, p\}$. Then C is not well-covered since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Recall that $p \nsim w$; otherwise we would have a forbidden subgraph by Claim 2.4/1.3.4.1. Also note that since $p \nsim q$ by birth, $p \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either n is adjacent to w, t is adjacent to w, n is adjacent to p, or p is adjacent to an additional vertex.

Suppose $n \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[n, q, p]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either n is adjacent to q, or n is adjacent to p. Suppose $n \sim q$. See Figure 50(a) for an illustration. To prevent $\{nr, nq, nw\}$ from forming a claw at n, either q is adjacent to w or r is adjacent to w. Suppose $q \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, $p \sim r$. To prevent $\{rt, rn, rp\}$ from forming a claw at r, we must have $p \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p]$ containing x, so that $V(C) = \{v, x, s, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim w$. Suppose $r \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $p \sim r$. But then $\{rp, rt, rn\}$ is a claw at r, since $p \nsim n$ by planarity. This contradicts the fact that G is claw-free, and so $r \nsim w$, thus $n \nsim q$. Suppose $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets

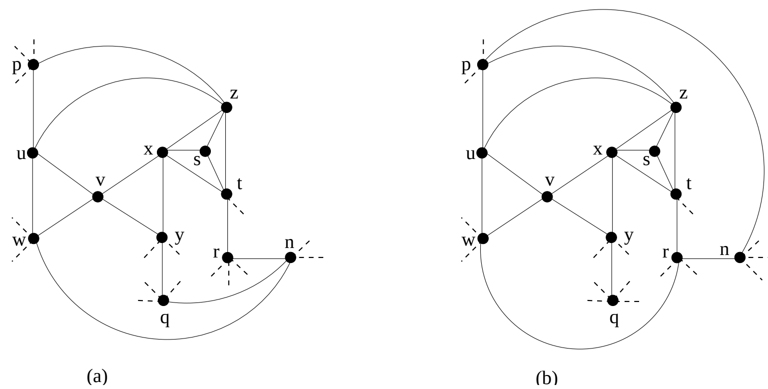


Figure 50: Proving that every vertex of degree four must lie on a K_4.

of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim p$, and therefore $n \nsim w$.

Suppose $t \sim w$. To prevent $\{ts, tr, tw\}$ from forming a claw at t, we must have $r \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, r]$ containing x, so that $V(C) = \{v, x, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $p \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[r, y]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $t \nsim w$. Thus we may assume that $d(t) = 4$.

Suppose $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[r, y, w]$ containing z, so that $V(C) = \{p, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{p, s\}$ are both maximal independent sets of C. Recall that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus by Lemma 2.2, either r is adjacent to w, p is adjacent to r, or p is adjacent to an additional vertex.

Suppose $w \sim r$. See Figure 50(b) for an illustration. Then $\{rt, rn, rw\}$ is a claw at r, contradicting the fact that G is claw-free. Hence $r \nsim w$.

Suppose $p \sim r$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must exist a path from w to the vertex set $\{y, r, q, p, n\}$ that does not pass through u or v. Thus either w is adjacent to q or w is adjacent to an additional vertex. Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Recall $p \nsim q$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim q$. Suppose w is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, m]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either m is adjacent to p or m is adjacent to q. Suppose $m \sim p$. To prevent $\{pu, pr, pm\}$ and $\{pu, pn, pm\}$ from forming claws at p, we must have $r \sim m$ and $n \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $m \nsim y$; otherwise $\{my, mw, mn\}$ would form a claw at m. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim p$. Suppose $m \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p, m]$ containing x, so that $V(C) = \{v, x, y, t, s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and s cannot grow, and $v \nsim s$. Recall that $m \nsim p$ by the preceding subcase. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $m \nsim q$, and so w is not adjacent to an additional vertex, and therefore $p \nsim r$.

Suppose p is adjacent to an additional vertex; call it m. Note that m could be interior or exterior to the $pnrtz$-face. To prevent $\{pu, pm, pn\}$ from forming a claw at p, we must have $m \sim n$. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must exist a path from w to the vertex set $\{y, r, q, p, n, m\}$ that does not pass through u or v. Note that $m \nsim w$ by birth. Thus either w is adjacent to q or w is

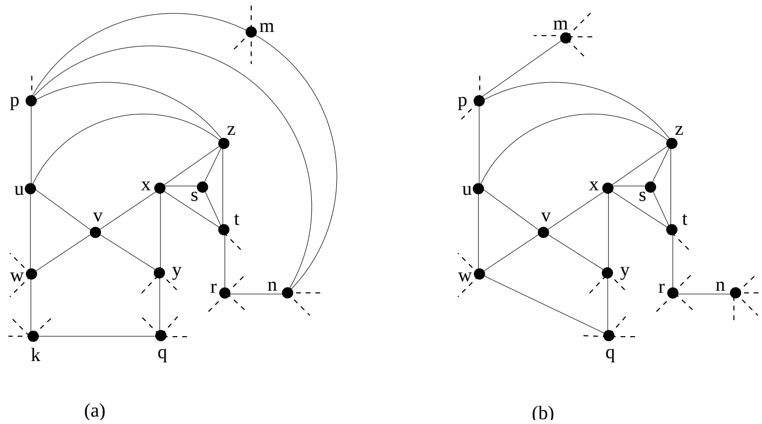


Figure 51: Proving that every vertex of degree four must lie on a K_4.

adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[q,p,r]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Note that $p \nsim r$ by a previous subcase (of the n adjacent to p subcase of the r adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.2.4.4.2.1.2.2). Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim q$.

Suppose w is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[q,p,r,k]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either k is adjacent to q, k is adjacent to p, or k is adjacent to r. Suppose $k \sim q$. See Figure 51(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[p,r,k]$ containing x, so that $V(C) = \{v,x,y,s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and s cannot grow, and $v \nsim s$. Thus if the set $\{p,r,k\}$ is independent, then by Lemma 2.3, G is not well-covered, a contradiction. Recall that $p \nsim r$ by a previous subcase (of the n adjacent to p subcase of the r adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.2.4.4.2.1.2.2). Note that since p is not adjacent to w or q, $p \nsim k$ by claw-freedom at k. Thus we must have $k \sim r$. To prevent $\{kw, kq, kr\}$ from forming a claw at k, we must have $r \sim w$. But then $\{rt, rk, rn\}$ is a claw at r, since t cannot grow and $n \nsim k$ by planarity. Hence $k \nsim q$. Suppose $k \sim p$. To prevent $\{pu, pn, pk\}$ and $\{pu, pm, pk\}$ from forming claws at p, we must have $n \sim k$ and $m \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, k]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1. Note that since $k \nsim q$ by the preceding subcase, $k \nsim y$ by claw-freedom at y. Also note that $k \nsim r$; otherwise $\{kw, kp, kr\}$ would form a claw at k, contradicting the fact that G is claw-free. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $k \nsim p$. Suppose $k \sim r$. To prevent $\{rt, rn, rk\}$ from forming a claw at r, we must have $n \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, k]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Recall that k is not adjacent to p or q by the two preceding subcases, and $p \nsim q$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $k \nsim r$, and so w is not adjacent to an additional vertex. Therefore p is not adjacent to an additional vertex (as a subcase of the $n \sim p$ case), and so $n \nsim p$.

Suppose p is adjacent to an additional vertex; call it m. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must exist a path from w to the vertex set $\{y, r, q, p, n, m\}$ that does not pass through u or v. Recall that w is not adjacent to t or n by previous subcases (of the r adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), and therefore $w \nsim r$ by claw-freedom at r. Note that $w \nsim m$ by birth, and therefore $w \nsim p$ by claw-freedom at p. Thus either w is adjacent to q or w is adjacent to an additional vertex.

Suppose $w \sim q$. See Figure 51(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[q,p,r]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Note that q is not adjacent to p or r by birth. Thus by Lemma 2.2, we must have $p \sim r$. But then $\{rp,rt,rn\}$ is a claw at r, since t cannot grow and $n \nsim p$ by the preceding subcase. This contradicts the fact that G is claw-free, and therefore $w \nsim q$.

Suppose w is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[q,p,r,k]$ containing x, so that $V(C) = \{v,x,s\}$. Then C is not well-covered since $\{x\}$ and $\{v,s\}$ are both maximal independent sets of C. Since $p \nsim n$ by a previous subcase (of the r adjacent to an additional vertex subcase of the z adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $p \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either k is adjacent to q, k is adjacent to p, or k is adjacent to r. Suppose $k \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[p,r,k]$ containing x, so that $V(C) = \{v,x,y,s\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, s and v cannot grow, and $v \nsim s$. Thus if the set $\{r,p,k\}$ is independent, then G is not well-covered by Lemma 2.3, a contradiction. Note that since p is not adjacent to q or w, $p \nsim k$ by claw-freedom at k. Hence we must have $k \sim r$. But then $\{kw,kq,kr\}$ is a claw at k, contradicting the fact that G is claw-free. Hence $k \nsim q$. Suppose $k \sim p$. To prevent $\{pu,pm,pk\}$ from forming a claw at p, we must have $k \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y,r,k]$ containing z, so that $V(C) = \{u,z,s\}$. Then C is not well-covered since $\{z\}$ and $\{u,s\}$ are both maximal independent sets of C. Note that since $k \nsim q$ by the preceding subcase, $k \nsim y$ by claw-freedom at y. Also since r is not adjacent to p or w, $r \nsim k$ by claw-freedom at k.

Thus by Lemma 1.4, G is not well-covered. A contradiction. Hence $k \nsim p$. Suppose $k \sim r$. To prevent $\{rt, rn, rk\}$ from forming a claw at r, we must have $n \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[q, p, k]$ containing x, so that $V(C) = \{v, x, t, s\}$. Then C is not well-covered since $\{x\}$ and $\{v, s\}$ are both maximal independent sets of C. Recall that k is not adjacent to either p or q by the two preceding subcases. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $k \nsim r$, and so w is not adjacent to an additional vertex. Thus p is not adjacent to an additional vertex, so r is not adjacent to an additional vertex, and therefore z is not adjacent to an additional vertex.

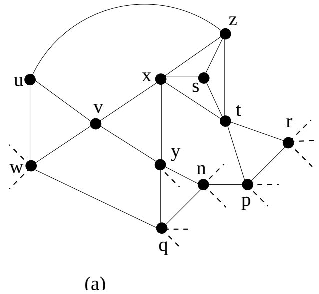
Note that since neither r nor q is adjacent to u by birth, neither r nor q is adjacent to z by claw-freedom at z. Also note that $z \nsim y$; otherwise we would have the forbidden subgraph shown in Figure 5(a) and centered at x. Thus either z is adjacent to w or $d(z) = 4$. This implies that $d(u) = 3$; otherwise an additional neighbor along with v and z would form a claw at u (recall that $u \nsim t$ by Claim 2.4/1.3.4.4.2.1.1).

Suppose t is adjacent to an additional vertex; call it p. To prevent $\{ts, tr, tp\}$ from forming a claw at t, we must have $p \sim r$. Since G is 3-connected, $\{y, t\}$ is not a 2-cut, and so there must exist a path from w to the set $\{r, q, p\}$ that does not pass through either y or t. Thus either w is adjacent to q, w is adjacent to r, w is adjacent to p, or w is adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r]$ containing x, so that $V(C) = \{x, y, z, s\}$. Then C is not well-covered since $\{x\}$ and $\{y, s\}$ are both maximal independent set of C. Note that since $r \nsim q$ by birth, $r \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, yq, yn\}$ from forming a claw at y, we must have $n \sim q$. Since G is 3-connected, $\{t\}$ is not a cut-vertex and so there must be a path from p and r to the vertex set $\{w, y, q, n\}$ that does not pass through t. Since $p \nsim q$ by birth, p is not adjacent to either y or w by claw-freedom at each of those vertices. Thus by Lemma 2.2, either p is adjacent to n, p is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose $p \sim n$. See Figure 52(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing x, so that $V(C) = \{x, y, z, s\}$. Then C is not well-covered since $\{x\}$ and $\{y, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it m. To prevent $\{yv, yq, ym\}$ and $\{yv, yn, ym\}$ from forming claws at y, we must have $q \sim m$ and $n \sim m$. Thus m must be in the yqn-face by planarity, and $d(y) = 5$. Since G is 3-connected, $\{t, p\}$ is not a 2-cut, and so there must be a path from r to the set $\{w, q, n\}$ that does not pass through either t or p. Recall that $r \nsim n$ by birth. Thus either n is adjacent to an additional vertex, or by claw-freedom w and q share an additional neighbor.

Suppose n is adjacent to an additional vertex; call it k. To prevent $\{nm, np, nk\}$ from forming a claw at n, we must have $k \sim p$. Now $d(n) = 5$ and so $d(p) = 4$ otherwise an additional neighbor along with t and n would create a claw at p (recall $p \nsim q$ by birth), contradicting the fact that G is claw-free. Either w is adjacent to k, w is adjacent to an additional vertex, or $d(w) = 3$. Suppose $w \sim k$. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $k \sim q$. Then $d(q) = 5$ and so $d(w) = 4$ otherwise an additional neighbor along with v and q would form a claw at w, contradicting the fact that G is claw-free. Similarly, $d(k) = 4$ otherwise an additional neighbor together with q and p would form a claw at k (note that $k \nsim r$ otherwise we would have a forbidden subgraph by Claim 2.4/1.3.4.1). But then $\{t, p\}$ is a 2-cut, separating r from the rest of the graph and contradicting the fact that G is claw-free. Hence $w \nsim k$. Suppose w is adjacent to an additional vertex; call it



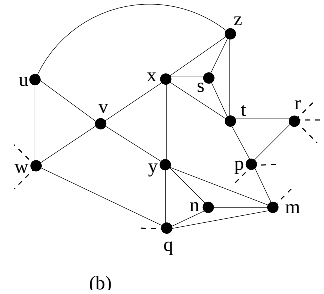


Figure 52: Proving that every vertex of degree four must lie on a K_4.

j. To prevent $\{wv, wq, wj\}$ from forming a claw at w, we must have $q \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Since neither y nor p may grow, $\{y, p, j\}$ is an independent set. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $d(w) = 3$. Then $d(q) = 4$; otherwise an additional neighbor along with w and y would form a claw at q, contradicting the fact that G is claw-free. But then $\{n, t\}$ is a 2-cut, separating the set $\{r, p, k\}$ from the rest of the graph and contradicting the fact that G is 3-connected. Therefore n is not adjacent to an additional vertex. Note that this implies that $d(p) = 3$; otherwise an additional neighbor along with t and n would form a claw at p, contradicting the fact that G is claw-free.

Suppose w and q share an additional neighbor, and call it k. Then $d(q) = 5$ and so $d(w) = 4$; otherwise an additional neighbor along with v and q would form a claw at w, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, k]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w and q do not share an additional neighbor, and therefore $p \nsim n$.

Suppose p is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[u, y, m]$ containing t, so that $V(C) = \{s, t, r\}$. Then C is not well-covered since $\{t\}$ and $\{s, r\}$ are both maximal independent sets of C. Note that $m \nsim r$ by by Claim 2.4/1.3.4.3 since p and r share t as a neighbor and t cannot grow. Thus by Lemma 2.2, either m is adjacent to y or r is adjacent to an additional vertex.

Suppose $m \sim y$. To prevent $\{yv, yq, ym\}$ and $\{yv, yn, ym\}$ from forming claws at y, we must have $m \sim q$ and $m \sim n$. See Figure 52(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, m]$ containing t, so that $V(C) = \{x, s, t, r\}$. Then C is not well-covered since $\{t\}$ and $\{s, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[u, n, k]$ containing t, so that $V(C) = \{x, s, t, p\}$. Then C is not well-covered since $\{t\}$ and $\{s, p\}$ are both maximal independent sets of C. Note that $k \nsim u$ by birth, and $k \nsim n$ by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim y$.

Suppose r is adjacent to an additional vertex; call it k. Note that $k \nsim p$ by Claim 2.4/1.3.4.3, since p and r share t as a neighbor and t cannot grow. Since G is 3-connected, $\{n, t\}$ is not a 2-cut, and so there must be a path from the vertex set $\{w, y, q\}$ to the vertex set $\{r, p, m, k\}$ that does not pass through either n or t. Note that if w is adjacent to an additional vertex, then q, must be adjacent to that vertex by claw-freedom; if y is adjacent to an additional vertex, then q must be adjacent to that vertex by claw-freedom; and if q is adjacent to an additional vertex, then either w or y must be adjacent to that vertex by claw-freedom. Also note that y is adjacent neither to m, by the preceding subcase, nor to k by birth. Thus either w and q are both adjacent to k, w and q are both adjacent to m, w and q share an additional neighbor, or y and q share an additional neighbor in the exterior face.

Suppose w and q are adjacent to k. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, k]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither w nor q is adjacent to k.

Suppose w and q are adjacent to m. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither w nor q is adjacent to m.

Suppose w and q share an additional neighbor; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus by Lemma 2.2, either j is adjacent to y, or j is adjacent to r. Suppose $j \sim y$. To prevent $\{yv, yn, yj\}$ from forming a claw at y, we must have $j \sim n$. See Figure 53(a) for an illustration. Now any additional neighbors adjacent to w, must be in the wqj-face by claw-freedom. But then $\{j, t\}$ is a 2-cut, separating $\{r, p, m, k\}$ from the rest of the graph and contradicting the fact that G is 3-connected. Hence $j \nsim y$. Suppose $j \sim r$. To prevent $\{rt, rk, rj\}$ from forming a claw at r, we must have $j \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, j]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $j \nsim p$; otherwise $\{pt, pm, pj\}$ would be a claw at p, since if p is adjacent to j, then $d(j) = 5$ and j may grow no further. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $j \nsim r$, and so w and q

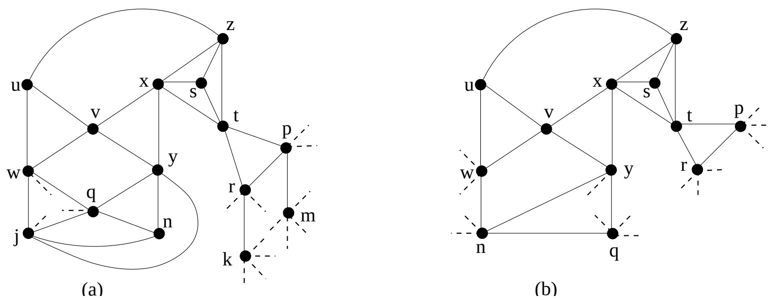


Figure 53: Proving that every vertex of degree four must lie on a K_4.

do not share an additional neighbor.

Suppose y and q share an additional neighbor in the exterior face; call it j. To prevent $\{yv, yn, yj\}$ from forming a claw at y, we must have $n \sim j$. Note that $d(w) = 4$ and w cannot grow, thus q cannot grow; otherwise an additional neighbor along with y and w would form a claw at q. But then $\{j, t\}$ is a 2-cut, separating $\{r, p, m, k\}$ from the rest of the graph and contradicting the fact that G is 3-connected. Hence q and y do not share an additional neighbor. Therefore r is not adjacent to an additional vertex (as a subcase of p adjacent to an additional vertex), and so p is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it m. Since $p \nsim n$ by a previous subcase (of the w adjacent to q subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $p \nsim y$ by claw-freedom at y. Since $p \nsim q$ by birth, p is not adjacent to w by claw-freedom at w. Also $p \nsim m$ by Claim 2.4/1.3.4.3, since p and r share the neighbor t and t cannot grow. But then, since the preceding subcase showed us that p is not adjacent to an additional vertex, $d(p) = 2$, which contradicts the fact that G is 3-connected. Hence r is not adjacent to an additional vertex and so $w \nsim q$.

Suppose $w \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $y \nsim r$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim r$.

Suppose $w \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that since $p \nsim q$ by birth, $y \nsim p$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim p$.

Suppose w is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, n]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus by Lemma 2.2, either n is adjacent to y or n is adjacent to r.

Suppose $n \sim y$. To prevent $\{yv, yq, yn\}$ from forming a claw at y, we must have $n \sim q$. See Figure 53(b) for an illustration. Since G is 3-connected, $\{q, t\}$ is not a 2-cut, and so there must exist a path from the vertex set $\{w, y, n\}$ to the vertex set $\{r, p\}$ that does not pass through either q or t. Since r is not adjacent to either y (by Claim 2.4/1.3.4.4.2.1.2.1) or w (by a previous subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $r \nsim n$ by claw-freedom at n. Since p is not adjacent to either y (by claw-freedom since $p \nsim q$) or w (by a previous subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2), $p \nsim n$ by claw-freedom at n. Thus by claw-freedom, either w and n share an additional neighbor in the exterior face, or y and n share an additional neighbor in the exterior face. Suppose w and n share an additional neighbor and call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either m is adjacent to y or m is adjacent to r. Suppose $m \sim y$. To prevent $\{yv, yq, ym\}$ from forming a claw at y, we must have $m \sim q$. Now any additional neighbors adjacent to w must be in the wnm-face by claw-freedom. But then $\{m, t\}$ is a 2-cut, separating r and p from the rest of the graph and contradicting the fact that G is 3-connected. Hence $m \nsim y$. Suppose $m \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, m]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall $m \nsim y$ by the preceding subcase. Thus by Lemma 2.2, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim r$, and so w and n do not share an additional neighbor. Note that this means $d(w) = 3$. Suppose y and n share an additional neighbor, and call it m. To prevent $\{yv, yq, ym\}$ from forming a claw at y, we must have $m \sim q$. Now $d(y) = 5$. But then $\{t, m\}$ is a 2-cut, separating p and r from the rest of the graph and contradicting the fact that G is 3-connected. Hence y and n do not share an additional neighbor, and so $n \nsim y$.

Suppose $n \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $n \nsim y$ by the preceding subcase. Thus by Lemma 2.2, we must have $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim r$, and so w is not adjacent to an additional vertex.

Therefore t is not adjacent to an additional vertex. Since $q \nsim r$ by birth, $q \nsim t$

by claw-freedom at t, and therefore $q \nsim y$ by claw-freedom at y. Thus since t is not adjacent to an additional vertex, either t is adjacent to w or $d(t) = 4$.

Suppose r is adjacent to an additional vertex; call it p. Since G is 3-connected, $\{y, r\}$ is not a 2-cut, and so there must be a path from w to q and p that passes through neither y nor r. Thus either w is adjacent to q, w is adjacent to p, or w is adjacent to an additional vertex.

Suppose $w \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r]$ containing x, so that $V(C) = \{x, y, z, s\}$. Then C is not well-covered since $\{x\}$ and $\{y, s\}$ are both maximal independent sets of C. Note that since $q \nsim r$ by birth, $r \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, yn, yq\}$ from forming a claw at y, we must have $n \sim q$. See Figure 54(a) for an illustration. Since G is 3-connected, $\{n, r\}$ is not a 2-cut, and so there must exist a path from the set $\{w, y, q\}$ to p that does not pass through either n or r. Since $q \nsim p$ by birth, p is not adjacent to w or y by claw-freedom at each vertex. Thus by claw-freedom either w and q share an additional neighbor in the exterior face, or y and q share an additional neighbor in the exterior face.

Suppose w and q share an additional neighbor; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $r \nsim y$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus either m is adjacent to y or m is adjacent to r. Suppose $m \sim y$. To prevent $\{yv, yn, ym\}$ from forming a claw at y, we must have $n \sim m$. Now any additional neighbors adjacent to w, must be in the wqm-face by claw-freedom. But then $\{m, r\}$ is a 2-cut, separating r and p from the rest of the graph and contradicting the fact that G is 3-connected. Hence $m \nsim y$. Suppose that $m \sim r$. To prevent $\{rt, rp, rm\}$ from forming a claw at

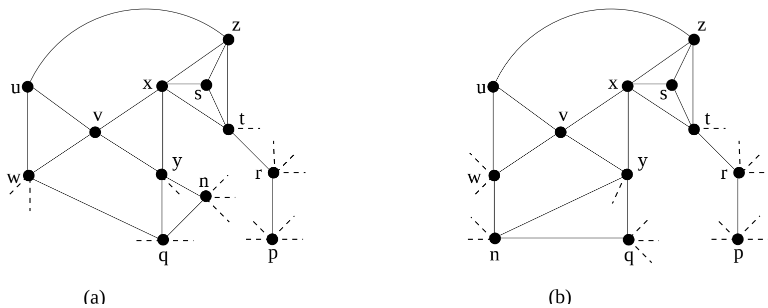


Figure 54: Proving that every vertex of degree four must lie on a K_4.

r, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $m \nsim y$ by the preceding subcase. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim r$, and so w and q do not share an additional neighbor. Thus either w is adjacent to t or $d(w) = 3$.

Suppose y and q share an additional neighbor in the exterior face and call it m. To prevent $\{yt, yn, ym\}$ from forming a claw at y, we must have $n \sim m$. But then $\{m, t\}$ is a 2-cut, separating r and p from the rest of the graph and contradicting the fact that G is 3-connected. Hence y and q do not share an additional neighbor, and so $w \nsim q$.

Suppose $w \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Since $p \nsim q$ by birth, $p \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $w \nsim p$.

Suppose w is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, n]$ containing z, so that $V(C) =$

$\{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $y \nsim r$ by Claim 2.4/1.3.4.4.2.1.2.1. Thus either n is adjacent to y, or n is adjacent to r.

Suppose $n \sim y$. To prevent $\{yv, yq, yn\}$ from forming a claw at y, we must have $q \sim n$. See Figure 54(b) for an illustration. Since G is 3-connected, $\{q, r\}$ is not a 2-cut and so there must exist a path from the set $\{w, y, n\}$ to p that does not pass through either q or r. Since $p \nsim q$ by birth, $p \nsim y$ by claw-freedom at y. Recall that $w \nsim p$ by a previous subcase (of the r adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.1.2.2). Thus $p \nsim n$ by claw-freedom at n. Hence by claw-freedom either w and n share an additional vertex in the exterior face, or y and n share an additional vertex in the exterior face. Suppose w and n share an additional vertex in the exterior face; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing z, so that $V(C) = \{u, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either m is adjacent to y, or m is adjacent to r. Suppose $m \sim y$. Now $d(y) = 5$ and so y cannot grow. To prevent $\{yv, yq, ym\}$ from forming a claw at y, we must have $m \sim q$. Any additional neighbors of w must be in the wnm-face; otherwise an additional neighbor together with v and n would be a claw at w, contradicting the fact that G is claw-free. But then $\{m, t\}$ is a 2-cut, separating p and r from the rest of the graph and contradicting the fact that G is 3-connected. Hence $m \nsim y$. Suppose $m \sim r$. To prevent $\{rt, rp, rm\}$ from forming a claw at r, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[m, y]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $m \nsim y$ by the preceding subcase. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim r$, and so w and n do not share an additional neighbor in the exterior face. Note that by 3-connectivity, w cannot be adjacent to an additional vertex in the $vwyn$-face.

Suppose $w \sim t$. Then by claw-freedom at t, $w \sim r$, and but then $\{wv, wt, wn\}$ is a claw at w since $d(t) = 5$ and so t cannot grow. Thus $d(w) = 3$. Suppose y and n share an additional neighbor in the exterior face; call it m. Now $d(y) = 5$. To prevent $\{yv, yq, ym\}$ from forming a claw at y, we must have $q \sim m$. Then $d(n) = 4$ and n cannot grow; otherwise an additional neighbor together with y and w would form a claw at n. But then $\{m, t\}$ is a 2-cut, separating r and p from the rest of the graph and contradicting the fact that G is 3-connected. Hence y and n do not share an additional neighbor, and therefore $n \nsim y$.

Suppose $n \sim r$. To prevent $\{rt, rp, rn\}$ from forming a claw at r, we must have $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[n, y]$ containing z, so that $V(C) = \{u, z, t, s\}$. Then C is not well-covered since $\{z\}$ and $\{u, s\}$ are both maximal independent sets of C. Recall that $n \nsim y$ by the preceding subcase. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim r$, and so w is not adjacent to an additional vertex. Therefore r is not adjacent to an additional vertex and so y is not adjacent to an additional vertex. Hence we have proved Claim 2.4/1.3.4.4.2.1.2.2. ■

By Claim 2.4/1.3.4.4.2.1.2.1, $r \nsim y$, and by Claim 2.4/1.3.4.4.2.1.2.2, y is not adjacent to an additional vertex. This is a contradiction and therefore t is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.4.2.1.2. ■

Claim 2.4/1.3.4.4.2.1.3: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.1.3: Suppose by way of contradiction, that y is adjacent to an additional vertex; call it r. Note that $y \nsim t$ otherwise we would have the forbidden subgraph shown in Figure 5(b) and centered at x. Also $u \nsim t$ by Claim 2.4/1.3.4.4.2.1.1. By Claim 2.4/1.3.4.4.2.1.2, t is not adjacent to an additional vertex (and this includes r since r was added after Claim 2.4/1.3.4.4.2.1.2 was proved). Since G is 3-connected and $\{x, z\}$ is not a 2-cut, there must be a path from t to the vertex

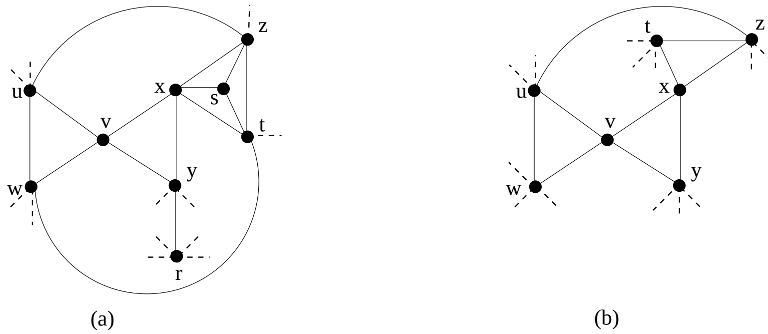


Figure 55: Proving that every vertex of degree four must lie on a K_4.

set $\{u, w, y, r\}$ that does not pass through either x or z. Thus we must have $t \sim w$. See Figure 55(a) for an illustration. Now $d(t) = 4$ and t cannot grow. Then w cannot have any additional neighbors in the $vwtxy$-face; otherwise this additional neighbor together with u and t would be a claw at w, contradicting the fact that G is claw-free. But then $\{x, v\}$ is a 2-cut, separating y and r from the rest of the graph, and contradicting the fact that G is 3-connected. Hence y is not adjacent to an additional vertex. ■

By Claim 2.4/1.3.4.4.2.1.1, $u \nsim t$. By Claim 2.4/1.3.4.4.2.1.2, t is not adjacent to an additional vertex. By Claim 2.4/1.3.4.4.2.1.3, y is not adjacent to an additional vertex. Together these claims are contradictory and therefore $d(x) < 5$ and we have proved Claim 2.4/1.3.4.4.2.1. ■

Note that by an argument similar to that used to prove Claim 2.4/1.3.4.4.2.1, we may now assume that $d(u) < 5$ as well.

Claim 2.4/1.3.4.4.2.2: The vertex x does not have degree four.

Proof of Claim 2.4/1.3.4.4.2.2: By way of contradiction, suppose $d(x) = 4$. Let the fourth neighbor of x be t. To prevent $\{xv, xz, xt\}$ from forming a claw at x, we must have $t \sim z$. Either t is interior to the $uvxz$-face, or exterior to that face.

Suppose that t is interior to the $uvxz$-face. See Figure 55(b) for an illustration. Since G is 3-connected, $\{x, z\}$ is not a 2-cut, and so there must be a path from t to u that does not pass through either x or z. Hence u must be adjacent to t or an additional vertex interior to the $uvxz$-face. Then u cannot be adjacent to an additional vertex exterior to the $uvxz$-face otherwise this additional exterior neighbor along with the additional interior neighbor and v would form a claw at u, contradicting the fact that G is claw-free. Since G is 3-connected, $\{u, x\}$ is not a 2-cut and so there must be a path from w and y to z that does not pass through either u or x. Now z is not adjacent to an additional vertex exterior to the $uvxz$-face; otherwise this exterior neighbor along with x and u would be a claw at z, contradicting the fact that G is claw-free. Also, $z \nsim y$ otherwise we would have a forbidden subgraph centered at x by Claim 2.4/1.3.4.1. Hence we must have $z \sim w$. Now w cannot have any neighbors in the $vwzxy$-face, since $w \nsim y$ by Claim 2.4/1.3.4.1, and if w is adjacent to an additional vertex, then this additional neighbor together with z and v is a claw at w, contradicting the fact that G is claw-free. But then $\{v, x\}$ is a 2-cut, separating y from the rest of the graph and contradicting the fact that G is 3-connected. Hence t must be exterior to the $uvxz$-face, and so by 3-connectivity, there are no additional vertices in the $uvxz$-face.

Call this semi-known subgraph S. Let C be the component of $S - N[u]$ containing x, so that $V(C) = \{x, y, t\}$. Then C is not well-covered since $\{x\}$ and $\{y, t\}$ are both maximal independent sets of C. Note that $y \nsim t$ otherwise we would have a forbidden subgraph centered at x, by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either u is adjacent to t, y is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.2.2.1: The vertex u is not adjacent to t.

Proof of Claim 2.4/1.3.4.4.2.2.1: Suppose by way of contradiction, that $u \sim t$.

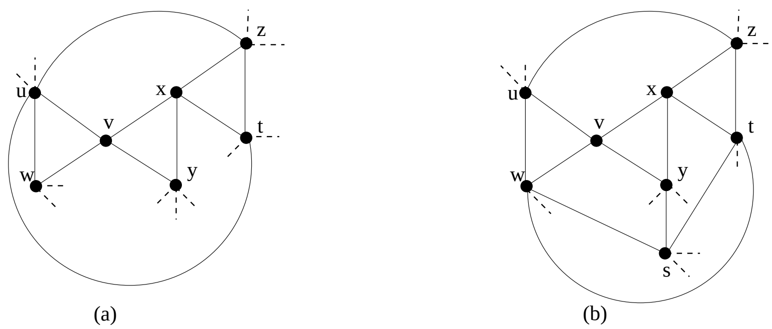


Figure 56: Proving that every vertex of degree four must lie on a K_4.

See Figure 56(a) for an illustration. Then z and t share two neighbors, x and u, so by Claim 2.4/1.3.4.3, we must have $x \sim u$. But x cannot grow, and therefore we have a contradiction and so $u \nsim t$. ∎

Claim 2.4/1.3.4.4.2.2.2: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.2.2: Suppose by way of contradiction that y is adjacent to an additional vertex; call it s. Since G is 3-connected, $\{u, x\}$ is not a 2-cut, and so there must exist a path from the vertex set $\{z, t\}$ to the vertex set $\{w, y, s\}$ that does not pass through either u or x. Thus either t is adjacent to w, z is adjacent to w, t is adjacent to s, z is adjacent to s, z is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.2.2.2.1: The vertex t is not adjacent to w.

Proof of Claim 2.4/1.3.4.4.2.2.2.1: Suppose, by way of contradiction, that $t \sim w$. Since G is 3-connected, y is not a cut-vertex and so there must exist a path from s to w and t that does not pass through y. Note that if w is adjacent to another vertex, then t must be adjacent to that vertex; otherwise the additional neighbor together with t and v would form a claw at w. Similarly, if t is adjacent to another vertex, then w must be adjacent to that vertex; otherwise the additional neighbor together with w and x would form a claw at t. Hence either t and w are both adjacent to s, or t and w share an additional neighbor in the $vwtxy$-face.

Suppose t and w are both adjacent to s. See Figure 56(b) for an illustration. Call this semi-known subgraph S. Let C be a component of $S - N[t]$, containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that if w is adjacent to any additional vertices, they must be in the wts-face; otherwise the additional neighbor together with v and s or t would form a claw at w. Similarly, if t is adjacent to any additional vertices, they must be in the wts-face; otherwise the additional neighbor together with x and s or w would form a claw at t. Hence there can be no additional vertices in the exterior face by 3-connectivity, and so u cannot grow (recall $u \nsim t$ by Claim 2.4/1.3.4.4.2.2.1). Also there can be no additional vertices in either the $vwsy$-face or the $xyst$-face, and so y cannot grow (recall y is not adjacent to either w or t by Claim 2.4/1.3.4.1). Hence C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Therefore neither w nor t is adjacent to s.

Suppose t and w share an additional neighbor in the $vwtxy$-face; call it r. Since G is 3-connected, y is not a cut-vertex and so there must be a path from s to the set $\{w, t, r\}$ that does not pass through y. Thus either r is adjacent to s, w and t share an additional neighbor interior to the $wrtxyv$-face, or r is adjacent to an additional vertex interior to the $wrtxyv$-face. Suppose $r \sim s$. Since G is 3-connected, $\{r, y\}$ is not a 2-cut, and so there must be a path from s to t that does not pass through either r or y. Note that t is not adjacent to an additional vertex in the $trsyx$-face; otherwise this additional vertex along with x and w would be a claw at t. Thus we must have $t \sim s$. But this contradicts the preceding subcase that proved that $t \nsim s$. Hence $r \nsim s$. Suppose w and t share an additional neighbor interior to the $wrtxyv$-face,and call it q. Then $d(w) = d(t) = 5$. But then $\{q, y\}$ is a 2-cut, separating s from the rest of the graph and contradicting the fact that G is 3-connected. Hence w

and t do not share a neighbor interior to the $wrtxyv$-face. Suppose r is adjacent to an additional vertex interior to the $wrtxyv$-face; call it q. But then $\{r, y\}$ is a 2-cut separating s and q from the rest of the graph, and contradicting the fact that G is 3-connected. Hence r is not adjacent to an additional vertex, and therefore t and w do not share an additional neighbor in the $vwtxy$-face.

Thus we have proved Claim 2.4/1.3.4.4.2.2.2.1, and $t \nsim w$. ■

Claim 2.4/1.3.4.4.2.2.2.2: The vertex z is not adjacent to w.

Proof of Claim 2.4/1.3.4.4.2.2.2.2: Suppose, by way of contradiction, that $z \sim w$. Then z cannot have any additional neighbors in the $vwztxy$-face; otherwise an additional such neighbor together with x and u would be a claw at z. (Note that $z \nsim y$; otherwise we would have a forbidden subgraph centered at x by Claim 2.4/1.3.4.2.) This implies that w cannot have an additional neighbor in the $vwztxy$-face; otherwise an additional such neighbor together with v and z would be a claw at w. (Note that $w \nsim y$; otherwise we would have a forbidden subgraph centered at v by Claim 2.4/1.3.4.1.) But then $\{y, t\}$ is a 2-cut, separating s from the rest of the graph and contradicting the fact that G is 3-connected. Hence $z \nsim w$. ■

Claim 2.4/1.3.4.4.2.2.2.3: The vertex t is not adjacent to s.

Proof of Claim 2.4/1.3.4.4.2.2.2.3: Suppose, by way of contradiction, that $t \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{y, u\}$ are both maximal independent sets of C. Recall that $t \nsim u$ by Claim 2.4/1.3.4.4.2.2.1, and $t \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.1. Thus by Lemma 2.2, either u is adjacent to an additional vertex, w is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it r. To prevent $\{uv, uz, ur\}$

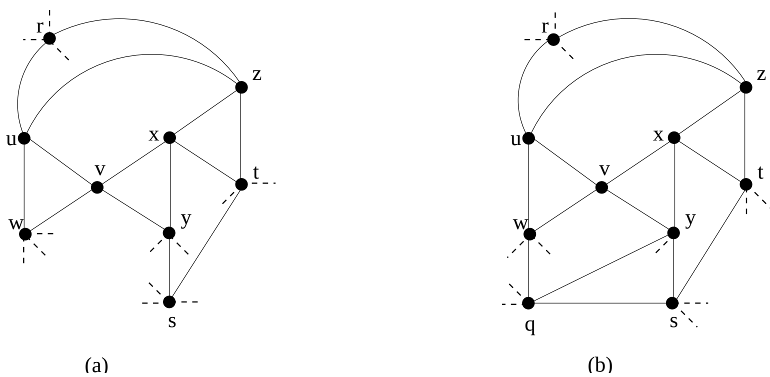


Figure 57: Proving that every vertex of degree four must lie on a K_4.

from forming a claw at u, we must have $r \sim z$. See Figure 57(a) for an illustration. Recall that by Claim 2.4/1.3.4.4.2.1, $d(u) < 5$, and so u cannot grow. This implies that $d(z) = 4$ as well, since $z \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.2, and if z were adjacent to an additional vertex, that vertex together with x and u would form a claw at z, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[r, t]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{y, u\}$ are both maximal independent sets of C. Note that $r \nsim t$ by birth. Thus by Lemma 2.2, either r is adjacent to y, w is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose $r \sim y$. To prevent $\{yv, ys, yr\}$ from forming a claw at y, we must have $s \sim r$. Note that $z \nsim s$; otherwise we would have a forbidden subgraph centered at r by Claim 2.4/1.3.4.1. Thus r cannot have any additional neighbors in the $ruwvy$-face; otherwise this additional neighbor together with z and s would form a claw at r, contradicting the fact that G is claw-free. Also y cannot have an additional neighbor in the $ruwvy$-face; otherwise this additional neighbor together with v and s would form a claw at y, contradicting the fact that G is claw-free. But then $\{u, v\}$ is a 2-cut, separating w from the rest of the graph and contradicting the fact that G is 3-connected. Hence $r \nsim y$.

Suppose w is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[q, z]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Since $q \nsim r$ by birth, $q \nsim z$ otherwise $\{zr, zq, zx\}$ would form a claw at z contradicting the fact that G is claw-free. Since $r \nsim t$ by birth and $r \nsim y$ by the preceding subcase, $r \nsim s$ by claw-freedom at s. Thus $s \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, either q is adjacent to y, q is adjacent to s, y is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $s \sim q$. See Figure 57(b) for an illustration. Since G is 3-connected, $\{u, z\}$ is not a 2-cut and so there must be a path from r to the set $\{w, q, s, t\}$ that does not pass through either u or z. Since $r \nsim q$, $r \nsim w$ by claw-freedom at w. Thus by claw-freedom either q and w share an additional exterior neighbor, or s and t share an additional exterior neighbor.

Suppose q and w share an additional exterior neighbor and call it p. Call this semi-known subgraph S. Let C be the component of $S - N[p, z]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $p \nsim u$; otherwise we would have a forbidden subgraph centered at w by Claim 2.4/1.3.4.1. Thus $p \nsim z$ by claw-freedom at z. Also note that $p \nsim s$; otherwise we would have a forbidden subgraph centered at q by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either y is adjacent to an additional vertex or s is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yq, yn\}$ and $\{yv, ys, yn\}$ from forming claws at y, we must have $q \sim n$ and $s \sim n$. Thus n must be in the ysq-face. Now $d(y) = 5$ and thus by Claim 2.4/1.2, $d(n) = 3$ and n cannot grow. Call this semi-known subgraph S. Let C be the component of

$S - N[p, z]$ containing y, so that $V(C) = \{v, y, s, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and n cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it n. To prevent $\{sy, st, sn\}$ from forming a claw at s, we must have $t \sim n$ since y is not adjacent to an additional vertex by the preceding subcase. Call this semi-known subgraph S. Let C be the component of $S - N[q, u]$ containing t, so that $V(C) = \{x, t, n\}$. Then C is not well-covered since $\{t\}$ and $\{x, n\}$ are both maximal independent sets of C. Note that $q \nsim u$; otherwise we would have a forbidden subgraph centered at w by Claim 2.4/1.3.4.2. Recall that $u \nsim t$ by Claim 2.4/1.3.4.4.2.2.1. Since $n \nsim z$ by birth, $n \nsim u$ by claw-freedom at u. Since $t \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.1 and $t \nsim y$ by Claim 2.4/1.3.4.1, $t \nsim q$ by claw-freedom at q. Finally, note that $q \nsim n$; otherwise we would have a forbidden subgraph centered at s by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either t is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it m. To prevent $\{tx, ts, tm\}$ and $\{tx, tn, tm\}$ from forming claws at t, we must have $s \sim m$ and $n \sim m$. Either m is interior to the tns-face or n is interior to the tsm-face. Since $d(t) = 5$, whichever vertex is interior cannot grow and has degree three by Lemma 2.4/1.2. Call this semi-known subgraph S. Let C be the component of $S - N[q, u]$ containing t, so that $V(C) = \{x, t, n, m\}$. Then every vertex of $C - t$ cannot grow, t, x and either m or n cannot grow, and x is adjacent to neither m nor n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex; note that this means $d(t) = 4$ and t cannot grow. Note that this also means that $d(s) = 4$ and s cannot grow, since an additional neighbor together with y and t would form a claw at s, contradicting the fact that G is claw-free.

Suppose n is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, m]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that $m \nsim q$ by birth, and C cannot grow. Thus by Lemma 2.2, we must have $m \sim r$. Since G is 3-connected, $\{q, w\}$ is not a 2-cut and so there must be a path from p to the set $\{r, n, m\}$ that does not pass through either q or w. Note that $p \nsim n$ by birth. Thus by claw-freedom either p is adjacent to both r and m, or p is adjacent to an additional vertex in the exterior face.

Suppose p is adjacent to both r and m. See Figure 58(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wp, wk\}$ and $\{wv, wq, wk\}$ from forming claws at w, we must have $p \sim k$ and $q \sim k$. Thus k must be in the wqp-face by planarity. Now w has degree five, and so by Claim 2.4/1.2, k cannot grow and hence has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, m]$ containing w, so that $V(C) = \{v, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{v, k\}$ are both maximal independent sets of C. Note that since $m \nsim u$ by birth, $m \nsim z$ by claw-freedom at z. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither r nor m is adjacent to p.

Suppose p is adjacent to an additional vertex in the exterior face; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, k]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either k is adjacent to r, k is adjacent to w, or w is adjacent to an additional vertex.

Suppose $k \sim r$. To prevent $\{rm, rz, rk\}$ from forming a claw at r, we must have $k \sim m$, since $k \nsim z$ by planarity. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, m]$ containing w, so that $V(C) = \{v, w, p\}$. Then C is not well-covered since $\{w\}$ and $\{v, p\}$ are both maximal independent sets of C. Recall $m \nsim p$ by the preceding subcase. Thus by Lemma 2.2, either w is adjacent to an additional vertex or p is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it j. To prevent $\{wv, wp, wj\}$ and $\{wv, wq, wj\}$ from forming claws at w, we must have $p \sim j$ and $q \sim j$. Thus by planarity, j must be interior to the wpq-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, j cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k]$ containing w, so that $V(C) = \{v, w, j\}$. Then C is not well-covered since $\{w\}$ and $\{v, j\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex. Suppose p is adjacent to an additional vertex; call it j. Note that j may be in either the $wurkp$-face or the $sqpkmn$-face. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, j]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow since w is not adjacent to an additional vertex, and note that $j \nsim s$ by birth. Thus by Lemma 2.2, we must have $j \sim r$, which forces j into the $wurkp$-face. But then $\{rz, rj, rm\}$ is a claw at r, contradicting the fact that G is claw-free. Thus p is not adjacent to an additional vertex and so $k \nsim r$.

Suppose $k \sim w$. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $k \sim q$. Now both w and q have degree five, and so by Claim 2.4/1.2, p cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, m]$ containing w, so that $V(C) = \{v, w, p, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and p cannot grow and $v \nsim p$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $k \nsim w$.

Suppose w is adjacent to an additional vertex; call it j. To prevent $\{wv, wp, wj\}$

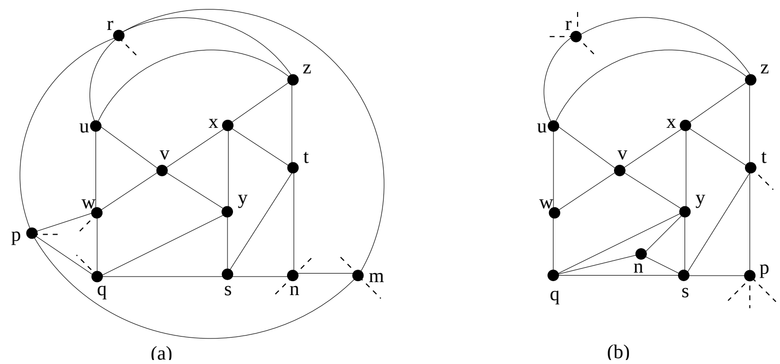


Figure 58: Proving that every vertex of degree four must lie on a K_4.

and $\{wv, wq, wj\}$ from forming claws at w, we must have $p \sim j$ and $q \sim j$. Now both w and q have degree five. Suppose j is exterior to the wpq-face. Then k is interior to the jpq-face (or the jpw-face), and so $\{j, p\}$ is a 2-cut, separating k from the rest of the graph and contradicting the fact that G is 3-connected. Hence j must be interior to the wqp-face. Then by Claim 2.4/1.2, j cannot grow and $d(j) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k]$ containing w, so that $V(C) = \{v, w, j\}$. Then C is not well-covered since $\{w\}$ and $\{v, j\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus w is not adjacent to an additional vertex, and therefore p is not adjacent to an additional vertex. Hence n is not adjacent to an additional vertex and so s is not adjacent to an additional vertex. Therefore, q and w do not share an external additional neighbor.

Suppose s and t share an additional external neighbor and call it p. Call this semi-known subgraph S. Let C be the component of $S - N[u, p]$ containing y, so that $V(C) = \{x, y, q\}$. Then C is not well-covered since $\{y\}$ and $\{x, q\}$ are both maximal independent sets of C. Note that $p \nsim z$; otherwise we would have a forbidden subgraph centered at t by Claim 2.4/1.3.4.1. Thus $p \nsim u$ by claw-freedom at u. Also note that q is not adjacent to an additional vertex in the exterior face; otherwise this additional neighbor together with w and y would be a claw at q. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yq, yn\}$ from forming claws at y, we must have $s \sim n$ and $q \sim n$. Thus n must be in the yqs-face by planarity. See Figure 58(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, p]$ containing y, so that $V(C) = \{x, y, q, n\}$. Now $d(y) = 5$ and so by Claim 2.4/1.2, n cannot grow and $d(n) = 3$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and n cannot grow, and $x \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence s and t do not share an additional external neighbor, and therefore $q \nsim y$.

Suppose $q \sim s$. But then $\{sy, st, sq\}$ is a claw at s, since $q \nsim t$ by birth, $q \nsim y$ by the preceding subcase, and $y \nsim t$ by Claim 2.4/1.3.4.1. This contradicts the fact that G is claw-free and hence $q \nsim s$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$ from forming a claw at y, we must have $p \sim s$. Suppose p is interior to the $xyst$-face. Then y is not adjacent to a vertex exterior to that face or the exterior neighbor together with p and v would form a claw at y, contradicting the fact that G is claw-free. Now t must be adjacent to p or another vertex interior to the $xyst$-face; otherwise $\{y, s\}$ would be a 2-cut separating p from the rest of the graph and contradicting the fact that G is 3-connected. Then t is not adjacent to a vertex exterior to the face; otherwise the exterior neighbor together with the interior neighbor and x would form a claw at t, contradicting the fact that G is claw-free. Then s may also not have an exterior neighbor or this exterior neighbor together with y and t would form a claw at s. But then $\{v, z\}$ is a 2-cut, separating $\{u, w, r, q\}$ from the rest of the graph, and contradicting the fact that G is claw-free. Hence p must be exterior to the $xyst$-face and there are no additional vertices in the $xyst$-face. See Figure 59(a) for an illustration. Call this semi-known subgraph S. Let C be the component of

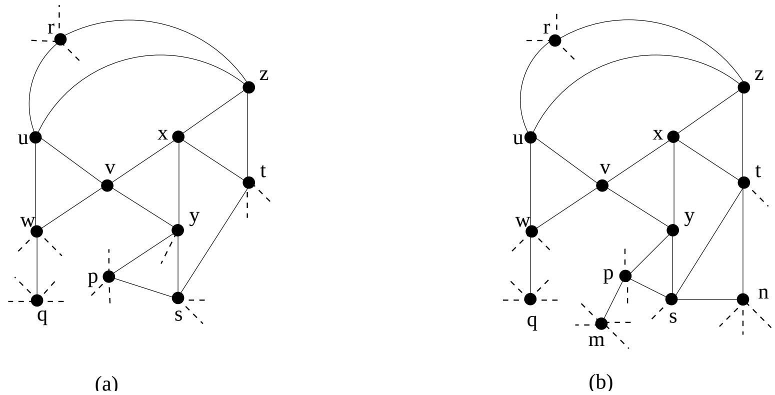


Figure 59: Proving that every vertex of degree four must lie on a K_4.

$S - N[q, z]$ containing y, so that $V(C) = \{v, y, p, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Since $q \nsim r$ by birth, $q \nsim z$ by claw-freedom at z. Note that p is not adjacent to either q or z by birth. Thus by Lemma 2.2, either y is adjacent to an additional vertex, s is adjacent to an additional vertex, or p is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yp, yn\}$ from forming claws at y, we must have $s \sim n$ and $p \sim n$. Either p is interior to the yns-face or n is interior to the ysp-face. Now $d(y) = 5$, thus by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[q, t, r]$ containing y, so that $V(C) = \{v, y, p, n\}$. Recall that q is adjacent to neither r or t by birth. Then every vertex of $C - y$ is adjacent to y, vertices y, v and one of p and n cannot grow, and v is adjacent to neither p nor n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex.

Suppose s is adjacent to an additional vertex; call it n. To prevent $\{sy, st, sn\}$ from forming a claw at s, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[q, t, r]$ containing y, so that $V(C) = \{v, y, p\}$. Thus by Lemma 2.2, either p is adjacent to r, or p is adjacent to an additional vertex.

Suppose $p \sim r$. Note that now u cannot have any neighbors in the $uvypr$-face; otherwise this neighbor together with z and v would form a claw at u. Thus both r and p must have neighbors in the $uvypr$-face; otherwise $\{w\}$ would be a cut-vertex, separating q from the rest of the graph, and contradicting the fact that G is 3-connected. Then neither r nor p may have neighbors in the $ztspr$-face or these neighbors together with their neighbors in the $uvypr$-face and u or y, respectively, would form claws at the respective vertices, contradicting the fact that G is claw-free. Also note that z cannot have any neighbors in the $ztspr$-face; otherwise this neighbor together with u and x would form a claw at z. But then $\{s,t\}$ is a 2-cut, separating n from the rest of the graph and contradicting the fact that G is 3-connected. Hence $p \nsim r$.

Suppose p is adjacent to an additional vertex; call it m. See Figure 59(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w,p,n]$ containing z, so that $V(C) = \{x,z,r\}$. Then C is not well-covered, since $\{z\}$ and $\{x,r\}$ are both maximal independent sets of C. Note that $n \nsim p$ otherwise we would have a forbidden subgraph centered at s by Claim 2.4/1.3.4.1. Since neither p nor n is adjacent to q by birth, neither p nor n is adjacent to w by claw-freedom at w. Also note that $n \nsim z$; otherwise we would have a forbidden subgraph centered at t by Claim 2.4/1.3.4.1. Recall that $p \nsim z$ by birth, and $p \nsim r$ by the preceding subcase. Also recall that z cannot grow. Thus by Lemma 2.2, either n is adjacent to r, or r is adjacent to an additional vertex.

Suppose $n \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[r,q,p]$ containing x, so that $V(C) = \{v,x,t\}$. Then C is not well-covered, since $\{x\}$ and $\{v,t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it k. To prevent $\{tx,ts,tk\}$ and $\{tx,tn,tk\}$

from forming claws at t, we must have $s \sim k$ and $n \sim k$. Thus by planarity, k must be interior to the tsn-face. Now $d(t) = 5$ and so, by Claim 2.4/1.2, k cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, p]$ containing t, so that $V(C) = \{x, t, k\}$. Then C is not well-covered, since $\{t\}$ and $\{x, k\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $n \nsim r$.

Suppose r is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[t, q, m, k]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Since $k \nsim n$ by birth, $k \nsim t$ by claw-freedom at t. Recall that t is not adjacent to m or q by birth of each. Thus by Lemma 2.2, either m is adjacent to k, or q is adjacent to k.

Suppose $m \sim k$. Since G is 3-connected, $\{w\}$ is not a cut-vertex and so there must be a path from q to the set $\{r, p, m, k\}$ that does not pass through w. Since q is not adjacent to m or r, $q \nsim k$ by claw-freedom at k. Thus q must be adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m, j]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $j \nsim t$ by planarity. Thus by Lemma 2.2, either j is adjacent to r, j is adjacent to m, j is adjacent to w, or w is adjacent to an additional vertex.

Suppose $j \sim r$. To prevent $\{rz, rk, rj\}$ from forming a claw at r, we must have $k \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[t, m, j]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, y, and u cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $j \nsim r$.

Suppose $j \sim m$. See Figure 60(a) for an illustration. To prevent $\{mp, mk, mj\}$

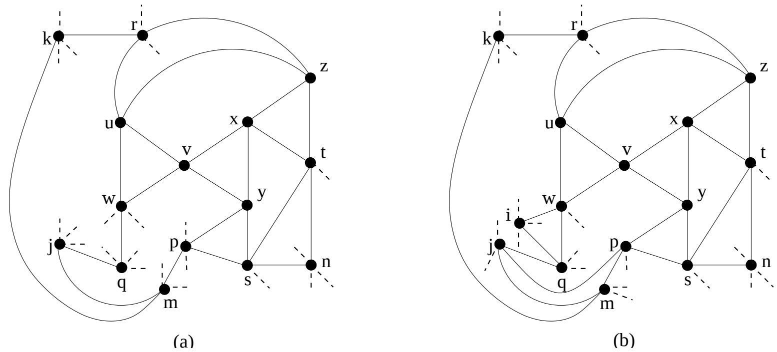


Figure 60: Proving that every vertex of degree four must lie on a K_4.

from forming a claw at m, either j is adjacent to p, or j is adjacent to k.

Suppose $j \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, j]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either j is adjacent to w, or w is adjacent to an additional vertex. Suppose $j \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[t, k, j]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Since k is not adjacent to either p or w by birth, $k \nsim j$ by claw-freedom at j. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $j \nsim w$. Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ from forming a claw at w, we must have $q \sim i$. See Figure 60(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, k, j, i]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $i \nsim j$ by birth and $i \nsim t$ by planarity. Thus by Lemma 2.2, either k is adjacent to i, or k is adjacent to j. Suppose $k \sim i$. To prevent $\{kr, ki, km\}$ from forming a claw at k, we must have $i \sim m$, since $i \nsim r$ by birth and $m \nsim r$ by birth. Now call this semi-known subgraph S. Let C be the component of $S - N[t, k, j]$ containing

v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u, and y cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $k \nsim i$. Suppose $k \sim j$. To prevent $\{jp, jq, jk\}$ from forming a claw at j, we must have $k \sim q$, since $k \nsim p$ by birth. But then $\{kq, kr, km\}$ is a claw at k, contradicting the fact that G is claw-free. Hence $j \nsim k$ as a subcase of w adjacent to an additional vertex. Thus w is not adjacent to an additional vertex and so $j \nsim p$.

Suppose $j \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, j]$ containing v, so that $V(C) = \{v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $j \nsim s$ by planarity. Thus by Lemma 2.2, either j is adjacent to w, or w is adjacent to an additional vertex. Suppose $j \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, j]$ containing y, so that $V(C) = \{v, y, p\}$. Then C is not well-covered since $\{y\}$ and $\{v, p\}$ are both maximal independent sets of C. Recall that $j \nsim r$ and $j \nsim p$ by the two preceding subcases. Note that $p \nsim t$ or we would have a forbidden subgraph centered at s, by Claim 2.4/1.3.4.2. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it i. To prevent $\{py, pm, pi\}$ from forming a claw at p, we must have $m \sim i$. Call this semi-known subgraph S. Let C be the component of $S - N[t, q, k, i]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $i \nsim t$ by birth, and $i \nsim k$; otherwise we would have a forbidden subgraph centered at m, by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either k is adjacent to q, or i is adjacent to q. Suppose $k \sim q$. But then $\{kr, kq, km\}$ is a claw at k, contradicting the fact that G is claw-free. Hence $k \nsim q$. Suppose $i \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[t, k, i]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, y and u cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $i \nsim q$ and so $j \nsim w$. Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ from forming a claw at w, we must have $i \sim q$. See Figure 61(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, j, i]$ containing y, so that $V(C) = \{v, y, p\}$. Then C is not well-covered since $\{y\}$ and $\{v, p\}$ are both maximal independent sets of C. Note that i is not adjacent to j or r by birth, and $i \nsim t$ by planarity. Thus by Lemma 2.2, either p is adjacent to i, or p is adjacent to an additional vertex. Suppose $p \sim i$. To prevent $\{py, pm, pi\}$ from forming a claw at p, we must have $i \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, k, i]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $k \nsim i$ by birth. Since $k \nsim n$ by birth, $k \nsim t$ by claw-freedom at t. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim i$. Suppose p is adjacent to an additional vertex; call it h. To prevent $\{py, pm, ph\}$ from forming a claw at p, we must have $h \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, k, i, h]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that h is not adjacent to either i or t by birth. Since h is not adjacent to j or r by birth, $h \nsim k$ by claw-freedom at k. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence p is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore $j \nsim k$, and thus $j \nsim m$.

Suppose $j \sim w$. See Figure 61(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k]$ containing w, so that $V(C) = \{v, w, q, j\}$. Then C is not well-covered since $\{w\}$ and $\{v, q\}$ are both maximal independent sets of C. Since $k \nsim n$ by birth, $k \nsim s$ by claw-freedom at s. Since neither q nor j are adjacent to r or m, neither q nor j are adjacent to k by claw-freedom at k. Thus by Lemma 2.2, either w is adjacent to an additional vertex, q is

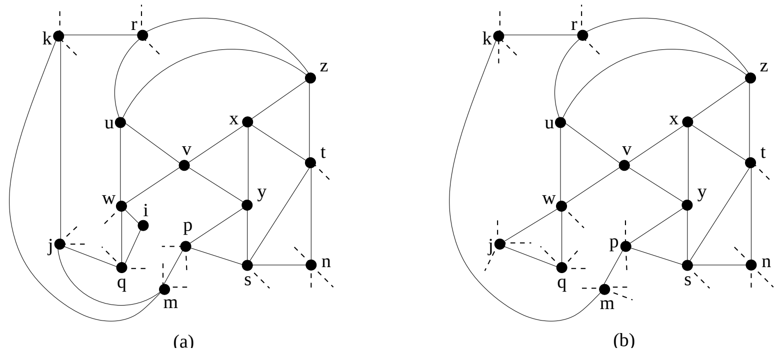


Figure 61: Proving that every vertex of degree four must lie on a K_4.

adjacent to an additional vertex, or j is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ and $\{wv, wj, wi\}$ from forming claws at w, we must have $i \sim q$ and $i \sim j$. Either i is interior to the wqj-face, j is interior to the wqi-face, or q is interior to the wji-face. Now $d(w) = 5$ and so whichever vertex is interior cannot grow and must have degree three, by Claim 2.4/1.2. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k]$ containing w, so that $V(C) = \{v, w, q, j, i\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of q, j or i cannot grow, and v is adjacent to none of q, j or i. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus w is not adjacent to an additional vertex and so $d(w) = 4$.

Suppose q is adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k, i]$ containing w, so that $V(C) = \{v, w, j\}$. Then C is not well-covered, since $\{w\}$ and $\{v, j\}$ are both maximal independent sets of C. Note that i is not adjacent to z, s and k by birth. Thus by Lemma 2.2, either i is adjacent to j, or j is adjacent to an additional vertex. Suppose $i \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m, i]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that since $i \nsim k$ by birth,

$i \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, we must have $i \sim m$. To prevent $\{mp, mk, mi\}$ from forming a claw at m, we must have $i \sim p$ (recall that $k \nsim p$ by birth). Call this semi-known subgraph S. Let C be the component of $S - N[t, r, i]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $i \nsim j$. Suppose j is adjacent to an additional vertex; call it h. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m, i, h]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $h \nsim i$ by birth, and $h \nsim t$ by planarity. Since $h \nsim k$ by birth, $h \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either m is adjacent to i or m is adjacent to h. Without loss of generality, since these cases are symmetric, suppose $m \sim i$. To prevent $\{mp, mk, mi\}$ from forming a claw at m, we must have $i \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, i, h]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither i nor h is adjacent to m, and so j is not adjacent to an additional vertex. Therefore q is not adjacent to an additional vertex.

Suppose j is adjacent to an additional vertex; call it i. But since q is not adjacent to r, p and m by birth, and therefore $q \nsim k$ by claw-freedom at k, and q is not adjacent to an additional vertex, $d(q) = 2$ which contradicts the fact that G is 3-connected. Hence j is not adjacent to an additional vertex and so $j \nsim w$.

Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ from forming a claw at w, we must have $i \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[z, q, n, k]$ containing y, so that $V(C) = \{v, y, p\}$. Then C is not well-covered, since $\{y\}$ and $\{v, p\}$ are both maximal independent sets of C. Since q is not adjacent to either r or m, $q \nsim k$ by claw-freedom at k. Recall that $k \nsim n$ by birth. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it h. To prevent $\{py, pm, ph\}$ from forming a claw at p, we must have $m \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[t, q, k, h]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that h is not adjacent to q or k by birth. Since $h \nsim n$ by birth, $h \nsim t$ by claw-freedom at t. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $m \nsim k$.

Suppose $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[t, m, k]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Note that $m \nsim t$ by birth, $m \nsim k$ by the preceding subcase, and since $k \nsim n$ by birth, $k \nsim t$ by claw-freedom at t. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim k$, and so r is not adjacent to an additional vertex. Therefore p is not adjacent to an additional vertex (as a subcase of s adjacent to an additional vertex), and so s is not adjacent to an additional vertex.

Suppose p is adjacent to an additional vertex; call it n. See Figure 62(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, q, n]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that n is not adjacent to q or z by birth. Since $q \nsim r$ by birth, $q \nsim z$ by claw-freedom at z. Recall that q is not adjacent to y or s by previous subcases (of the w adjacent to an additional vertex subcase of the u adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.3). Since r is not adjacent to y or t, $r \nsim s$ by claw-freedom at s, and $s \nsim z$ by claw-freedom at z, and therefore $z \nsim y$ by claw-freedom at y. Since y and

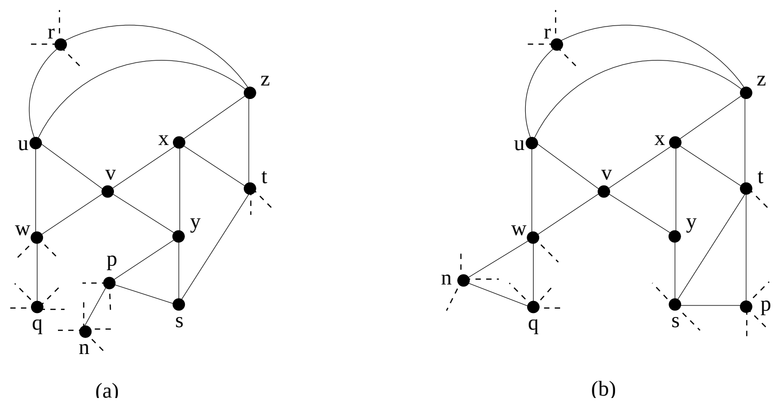


Figure 62: Proving that every vertex of degree four must lie on a K_4.

s are not adjacent to additional vertices, C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence p is not adjacent to an additional vertex and so y is not adjacent to an additional vertex (as a subcase of w adjacent to an additional vertex). Note that this implies that $d(y) = 3$ since y is not adjacent to any vertex in the set $\{u, w, z, t, r, q\}$ either by previous subcases within the subcases of Claim 2.4/1.3.4.4.2.2.2.3 or forbidden subgraphs.

Suppose s is adjacent to an additional vertex; call it p. To prevent $\{sy, st, sp\}$ from forming a claw at s, we must have $p \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s]$ containing w, so that $V(C) = \{v, w, q\}$. Then C is not well-covered, since $\{w\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that $s \nsim z$; otherwise we would have a forbidden subgraph centered at t by Claim 2.4/1.3.4.2. Since $q \nsim s$ by a previous subcase (of the w adjacent to an additional vertex subcase of the u adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.3), $q \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wq, wn\}$ from forming a claw at w, we must have $q \sim n$. See Figure 62(b) for an illustration.

Call this semi-known subgraph S. Let C be the component of $S - N[z, s]$ containing w, so that $V(C) = \{v, w, q, n\}$. Then C is not well-covered, since $\{w\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that n is not adjacent to s or z by birth. Thus by Lemma 2.2, either w is adjacent to an additional vertex, q is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wq, wm\}$ and $\{wv, wn, wm\}$ from forming claws at w, we must have $m \sim q$ and $m \sim n$. Either m is interior to the wqn-face, n is interior to the wqm-face, or q is interior to the wnm-face. Now $d(q) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, s]$ containing w, so that $V(C) = \{v, w, q, n, m\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of q, n and m cannot grow, and v is adjacent to none of q, n and m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $d(w) = 4$.

Suppose q is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, m]$ containing w, so that $V(C) = \{v, w, n\}$. Then C is not well-covered since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that m is not adjacent to s or z by birth. Also note that $m \nsim n$ by Claim 2.4/1.3.4.3. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m, k]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $k \nsim m$ by birth, and since $k \nsim s$ by birth, $k \nsim t$ by claw-freedom at t. Similarly, since $m \nsim s$ by birth, $m \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either k is adjacent to r or m is adjacent to r. Suppose, without loss of generality, that $k \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t, m, k]$

containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, w and y cannot grow, and $w \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence neither k nor m is adjacent to r, and so q is not adjacent to an additional vertex.

Suppose n is adjacent to an additional vertex; call it m. Note that q is not adjacent to any of $\{t, r, p\}$ by birth, and $q \nsim s$ by a previous subcase (of the w adjacent to an additional vertex subcase of the u adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.3). Thus if q is not adjacent to an additional vertex, then $d(q) = 2$, contradicting the fact that G is 3-connected. Hence n is not adjacent to an additional vertex and so w is not adjacent to an additional vertex and so $d(w) = 3$.

Suppose q is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, n]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Since $n \nsim s$ by birth, $n \nsim t$ by claw-freedom at t. Recall that $r \nsim t$ by birth. Thus by Lemma 2.2, we must have $n \sim r$. See Figure 63(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, n]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, w and y cannot grow, and $w \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so s is not adjacent to an additional vertex. Therefore w is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $q \sim s$. See Figure 63(b) for an illustration. Recall that w is not adjacent to t or z by Claims 2.4/1.3.4.4.2.2.2.1 and 2.4/1.3.4.4.2.2.2.2 respectively. Note that w is not adjacent to y or r; otherwise

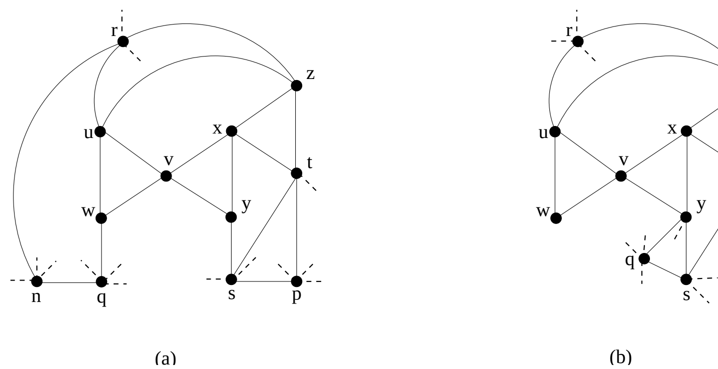


Figure 63: Proving that every vertex of degree four must lie on a K_4.

we would have forbidden subgraph centered at v and u, respectively, by Claim 2.4/1.3.4.1. Since w is not adjacent to either t or y, $w \nsim s$ by claw-freedom at s (note that $y \nsim t$ by Claim 2.4/1.3.4.1). But then since w is not adjacent to an additional vertex (which includes q) by the preceding subcase, $d(w) = 2$, contradicting the fact that G is 3-connected. Hence y is not adjacent to an additional vertex and so u is not adjacent to an additional vertex.

Note that $u \nsim t$ by Claim 2.4/1.3.4.4.2.2.1, $u \nsim y$ by Claim 2.4/1.3.4.1, and $u \nsim s$ by birth. Thus since u is not adjacent to an additional vertex, $d(u) = 3$. This implies that $d(z) = 3$ as well; otherwise if z were adjacent to an additional vertex, the additional neighbor together with u and x would form a claw at z (recall that $z \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.2).

Suppose w is adjacent to an additional vertex; call it r. See Figure 64(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, r]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that $r \nsim t$ by birth. Thus by Lemma 2.3, either r is adjacent to y, or y is adjacent to an additional vertex.

Suppose $r \sim y$. To prevent $\{yv, ys, yr\}$ from forming a claw at y, we must have

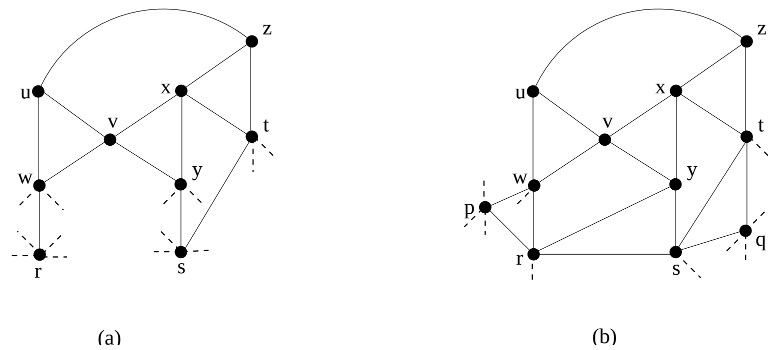


Figure 64: Proving that every vertex of degree four must lie on a K_4.

$s \sim r$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(j). Since G is not a graph from the Figure 2, this graph must grow. Since there are no possible additional edges between known vertices, there must be additional vertices. Note that by symmetry, since $d(x) = 4$, $d(y) = 4$ as well. Thus, by claw-freedom, either s and t are adjacent to an additional vertex, or w and r are adjacent to an additional vertex. Note that if only one of these options occurs, then the two vertices adjacent to the additional vertex form a 2-cut, separating that additional vertex from the rest of the graph, and contradicting the fact that G is 3-connected. Hence both options must occur. Either the four vertices share an additional neighbor, or s and t have a distinct additional neighbor from w and r. Suppose all four vertices share an additional neighbor, and call it q. Then we have a forbidden subgraph centered at s, by Claim 1.3.4.1. Suppose s and t have a distinct additional neighbor from w and r, and suppose that the neighbors are q and p, respectively. See Figure 64(b) for an illustration. Call this semi-known subgraph S. Let C be a component of $S - N[t, p]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim t$ since t does not share an additional neighbor with w and r. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence the graph cannot grow and so $r \nsim y$.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, yq, ys\}$ from forming a claw at y, we must have $q \sim s$. Suppose q is in the $yxts$-face. Then y is not adjacent to an additional vertex exterior to the $yxts$-face; otherwise the additional neighbor together with v and q would be a claw at y, contradicting the fact that G is claw-free. Then t must be adjacent to a vertex interior to the $yxts$-face; otherwise $\{y, s\}$ is a 2-cut, contradicting the fact that G is claw-free. Thus t also cannot be adjacent to an additional exterior vertex; otherwise the exterior neighbor together with the interior neighbor and x would form a claw at t, contradicting the fact that G is claw-free. But then $\{w, s\}$ is a 2-cut, separating r from the rest of the graph, and contradicting the fact that G is claw-free. Hence q must be exterior to the $yxts$-face. Since G is 3-connected, $\{y, s\}$ is not a 2-cut, and so there must be a path from q to the set $\{w, t, r\}$ that does not pass through either y or s. Since q is not adjacent to r or t by birth, and therefore is not adjacent to w by claw-freedom at w, q must be adjacent to an additional vertex exterior to the yqs-face; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, p]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall $r \nsim y$ by the preceding subcase. Thus by Lemma 2.2, either p is adjacent to r, p is adjacent to t, p is adjacent to y, or y is adjacent to an additional vertex.

Suppose $p \sim r$. See Figure 65(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either p is adjacent to t, p is adjacent to y, p is adjacent to w, y is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Suppose $p \sim t$. To prevent $\{tx, ts, tp\}$ from forming a claw at t, we must have

$p \sim s$. But then $\{pr, pq, pt\}$ is a claw at p, contradicting the fact that G is claw-free. Thus $p \nsim t$.

Suppose $p \sim y$. To prevent $\{yv, ys, yp\}$ from forming a claw at y, we must have $p \sim s$. Now $d(y) = 5$ and so by Claim 2.4/1.2, q cannot grow and $d(q) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s, q\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and q cannot grow, and $v \nsim q$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $p \nsim y$.

Suppose $p \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Recall that p is not adjacent to t or y by the two preceding subcases. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yq, yn\}$ from forming claws at y, we must have $s \sim n$ and $q \sim n$. Thus, by planarity, n must be in the yqs-face. Now $d(y) = 5$, and so by Claim 2.4/1.2, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, p]$ containing y, so that $V(C) = \{v, y, s, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and n cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus $p \nsim w$.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ and $\{yv, yq, yn\}$ from forming claws at y, we must have $s \sim n$ and $q \sim n$. Thus, by planarity, n must be in the yqs-face. Now $d(y) = 5$, and so by Claim 2.4/1.2, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s, q, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and n cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus y is not adjacent to an

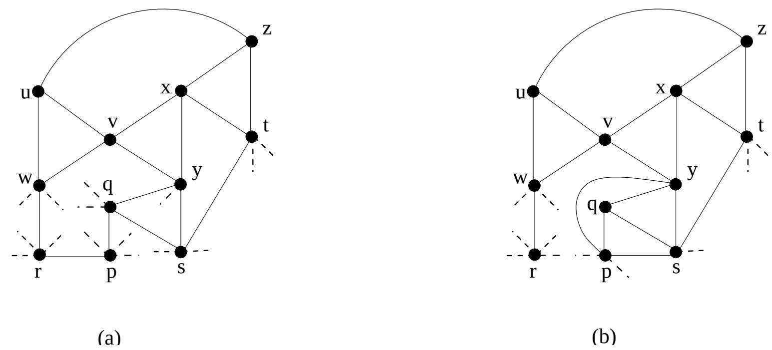


Figure 65: Proving that every vertex of degree four must lie on a K_4.

additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wr, wn\}$ from forming a claw at w, we must have $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that n is not adjacent to p or t by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $p \nsim r$.

Suppose $p \sim t$. To prevent $\{tx, ts, tp\}$ from forming a claw at t, we must have $p \sim s$. But then we have a forbidden subgraph centered at s, by Claim 2.4/1.3.4.1. Thus $p \nsim t$.

Suppose $p \sim y$. To prevent $\{yv, ys, yp\}$ from forming a claw at y, we must have $s \sim p$. Now $d(y) = 5$, and so by Claim 2.4/1.2, q cannot grow and $d(q) = 3$. See Figure 65(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and q cannot grow, and $v \nsim q$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $p \nsim y$.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, yq, yn\}$ and $\{yv, ys, yn\}$ from forming claws at y, we must have $q \sim n$ and $s \sim n$. Suppose n is exterior to the yqs-face. Then s cannot have any additional neighbors interior to the sqn-face; otherwise this interior neighbor together with y and t would be a claw at s. But then $\{q, n\}$ is a 2-cut, separating p from the rest of the graph and contradicting the fact that G is claw-free. Hence n must be interior to the yqs-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, n cannot grow and $d(n) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing y, so that $V(C) = \{v, y, s, n\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and n cannot grow, and $v \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex (a fifth neighbor), and so y is not adjacent to an additional vertex (a fourth neighbor, and as a subcase of w adjacent to an additional vertex). Thus w is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it r. To prevent $\{yv, ys, yr\}$ from forming a claw at y, we must have $s \sim r$. Recall that w is not adjacent to t or z by Claim 2.4/1.3.4.4.2.2.2.1 and Claim 2.4/1.3.4.4.2.2.2.2, respectively. Note that $w \nsim y$; otherwise we would have a forbidden subgraph centered at v, by Claim 2.4/1.3.4.1. Thus $w \nsim s$; otherwise we would have a claw at s with w, y and t, since $y \nsim t$ by Claim 2.4/1.3.4.1. But then since w is not adjacent to an additional vertex (and this includes r), $d(w) = 2$, which contradicts the fact that G is 3-connected. Hence y is not adjacent to an additional vertex and so $t \nsim s$. ■

Claim 2.4/1.3.4.4.2.2.2.4: The vertex z is not adjacent to s.

Proof of Claim 2.4/1.3.4.4.2.2.2.4: Suppose, by way of contradiction, that $z \sim s$. But then $\{zx, zu, zs\}$ is a claw at z, since x cannot grow and $s \nsim u$ by birth. Hence $z \nsim s$. ■

Claim 2.4/1.3.4.4.2.2.2.5: The vertex z is not adjacent to an additional vertex.

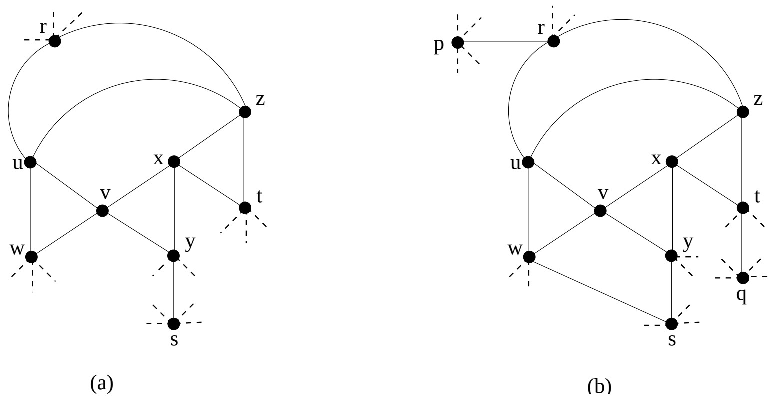


Figure 66: Proving that every vertex of degree four must lie on a K_4.

Proof of Claim 2.4/1.3.4.4.2.2.2.5: Suppose, by way of contradiction, that z is adjacent to an additional vertex; call it r. To prevent $\{zx, zu, sr\}$ from forming a claw at z, we must have $r \sim u$. See Figure 66(a) for an illustration. Recall by Claim 2.4/1.3.4.4.2.1, we may assume $d(u) < 5$. Thus here $d(u) = 4$ and u cannot grow. Then $d(z) = 4$ and z cannot grow, since $z \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.2, $z \nsim y$ by Claim 2.4/1.3.4.2, and if z were adjacent to an additional vertex, then that vertex together with u and x would form a claw at z. Call this semi-known subgraph S. Let C be the component of $S - N[w, y]$ containing z, so that $V(C) = \{z, t, r\}$. Then C is not well-covered since $\{z\}$ and $\{t, r\}$ are both maximal independent sets of C. Note that r is not adjacent to t; otherwise we would have a forbidden subgraph centered at z by Claim 2.4/1.3.4.1. Also note that r is not adjacent to w; otherwise we would have a forbidden subgraph centered at u by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either r is adjacent to y, t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose $r \sim y$. To prevent $\{yv, ys, yr\}$ from forming a claw at y, we must have $r \sim s$. Suppose s is exterior to the $vwury$-face. Now y cannot have any additional neighbors interior to the $vwury$-face; otherwise this neighbor together with v and s would form a claw at y, contradicting the fact that G is claw-free. Also r cannot have any additional neighbors interior to the $vwury$-face; otherwise this neighbor together with z and s would form a claw at r, contradicting the fact that G is claw-free. But then $\{u, v\}$ is a 2-cut, separating w from the rest of the graph and contradicting the fact that G is claw-free. Thus, s must be interior to the $vwury$-face. Now y cannot have any additional neighbors exterior to the $vwury$-face; otherwise this neighbor together with v and s would form a claw at y, contradicting the fact that G is claw-free. Also r cannot have any additional neighbors exterior to the $vwury$-face; otherwise this neighbor together with z and s would form a claw at r, contradicting the fact that G is claw-free. But then $\{x, z\}$ is a 2-cut, separating t from the rest of the graph and contradicting the fact that G is claw-free. Hence $r \nsim y$.

Suppose t is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[s, t]$ containing u, so that $V(C) = \{u, v, w, r\}$. Then C is not well-covered since $\{u\}$ and $\{r, v\}$ are both maximal independent sets of C. Recall that by Claim 2.4/1.3.4.4.2.2.2.3, $s \nsim t$. Thus by Lemma 2.2, either w is adjacent to s, r is adjacent to s, w is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose $s \sim w$. Call this semi-known subgraph S. Let C be the component of $S - N[s, t]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{r, v\}$ are both maximal independent sets of C. Note that since r is adjacent to neither w nor y (by Claim 2.4/1.3.4.1 and by the preceding subcase of Claim 2.4/1.3.4.4.2.2.2.5 respectively), $r \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it p. See Figure 66(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, p]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since p is not adjacent to s by birth, p is not adjacent to either y or w by claw-freedom at those vertices.

Recall that q is not adjacent to y or w by birth. Thus either p is adjacent to q, or w is adjacent to an additional vertex.

Suppose $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing u, so that $V(C) = \{u, w, z, r\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either q is adjacent to r, vertices r is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Suppose $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wv, ws, wn\}$ from forming a claw at w, we must have $n \sim s$. Note that n could be interior or exterior to the $vwsy$-face. One of the two possible cases is shown in Figure 67(a). Call this semi-known subgraph S. Let C be the component of $S - N[r, n]$ containing x, so that $V(C) = \{v, x, y, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that since $n \nsim q$ by birth, n is adjacent to neither r nor t by claw-freedom at those vertices. Also note that $n \nsim y$ by birth. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it m. To prevent $\{yv, ys, ym\}$ from forming a claw at y, we must have $m \sim s$. Note that m could be either interior or exterior to the $vwsy$-face. Call this semi-known subgraph S. Let C be the component of $S - N[r, n, m]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that m is adjacent to neither n nor r by birth. Thus by Lemma 2.2, either m is adjacent to t, or t is adjacent to an additional vertex.

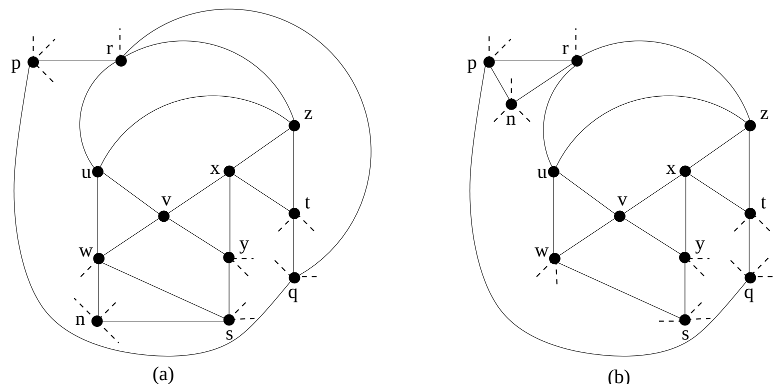


Figure 67: Proving that every vertex of degree four must lie on a K_4.

Suppose $m \sim t$. Then m must be exterior to the $vwsy$-face. To prevent $\{tx, tq, tm\}$ from forming a claw at t, we must have $m \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, vertex t must be adjacent to an additional vertex; call it k. To prevent $\{tx, tq, tk\}$ and $\{tx, tm, tk\}$ from forming claws at t, we must have $q \sim k$ and $m \sim k$. Thus by planarity k must be interior to the tqm-face. Now $d(t) = 5$, and so by Claim 2.4/1.2 vertex k cannot grow and so $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[v, s, p]$ containing t, so that $V(C) = \{z, t, k\}$. Then C is not well-covered since $\{t\}$ and $\{z, k\}$ are both maximal independent sets of C. Recall that $p \nsim s$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim t$.

Suppose t is adjacent to an additional vertex; call it k. To prevent $\{tx, tq, tk\}$ from forming a claw at t, we must have $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, m]$ containing t, so that $V(C) = \{x, t, k\}$. Then C is not well-covered since $\{t\}$ and $\{x, k\}$ are both maximal independent sets of C. Note that k is not adjacent to m, n or r by birth, and so $k \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either t is adjacent to an additional vertex, k is adjacent to an additional vertex. Suppose t is adjacent to an additional vertex; call it j. To prevent $\{tx, tq, tj\}$ and $\{tx, tk, tj\}$ from forming claws at t, we must have $q \sim j$ and $k \sim j$. By planarity, either k is interior to the tqj-face, or j is interior to the tqk-face. Now $d(t) = d(q) = 5$ and so by Claim 2.4/1.2, whichever of k or j is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, m]$ containing t, so that $V(C) = \{x, t, k, j\}$. Then every vertex of $C - t$ is adjacent to t, vertices t, x and either k or j cannot grow, and x is adjacent to neither k nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus t is not adjacent to an additional vertex and so $d(t) = 4$. Suppose k is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, j]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that $j \nsim r$ by birth. Also note that since j is not adjacent to either m or w by birth, $j \nsim s$ by claw-freedom at s. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence k is not adjacent to an additional vertex, and therefore t is not adjacent to an additional vertex.

Therefore y is not adjacent to an additional vertex. Note that this implies that y cannot grow and $d(y) = 3$.

Suppose t is adjacent to an additional vertex; call it m. To prevent $\{tx, tq, tm\}$ from forming a claw at t, we must have $m \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, m]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{y, z\}$ are both maximal independent sets of C. Since $m \nsim n$ by birth, $m \nsim w$ by claw-freedom at w. Note that $m \nsim p$; otherwise we would have a forbidden subgraph centered at q, by Claim 2.4/1.3.4.1. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex and so $q \nsim r$.

Suppose r is adjacent to an additional vertex; call it n. To prevent $\{ru, rp, rn\}$ from forming a claw at r, we must have $p \sim n$. Note that n could be either interior or exterior to the $uwsyxtqpr$-face. One of the two possible cases is shown in Figure 67(b). Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that n is adjacent to neither q nor y by birth. Thus by Lemma 2.2, either w is adjacent to n, or w is adjacent to an additional vertex.

Suppose $w \sim n$. To prevent $\{wv, ws, wn\}$ from forming a claw at w, we must have $n \sim s$. Note that this forces n to lie interior to the $uwsyxtqpr$-face. Call this semi-known subgraph S. Let C be the component of $S - N[t, n]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Since $n \nsim q$ by birth, $n \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, vertex y must be adjacent to an additional vertex; call it m. To prevent $\{yv, ys, ym\}$ from forming a claw at y, we must have $s \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, m]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that $m \nsim r$ by birth. Thus by Lemma 2.2, either m is adjacent to q, or w is adjacent to an additional vertex.

Suppose $m \sim q$. To prevent $\{qt, qp, qm\}$ from forming a claw at q, we must have $m \sim p$, since neither p nor m is adjacent to t by their births. Now r cannot have any additional neighbors in the $ztqpr$-face; otherwise this additional neighbor together with u and n would form a claw at r, contradicting the fact that G is claw-free. Also p cannot have any additional neighbors in the $ztqpr$-face; otherwise this additional neighbor together with n and m would form a claw at p, contradicting the fact that G is claw-free. Now y cannot have any additional neighbors in the

$yxtqm$-face; otherwise this additional neighbor together with v and s would form a claw at y, contradicting the fact that G is claw-free. Also m cannot have any additional neighbors in the $yxtqm$-face; otherwise this additional neighbor together with s and p would form a claw at m, contradicting the fact that G is claw-free. Hence by 3-connectivity, t cannot grow (it has no face to grow into), and so $d(t) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing x, so that $V(C) = \{x, y, z, t\}$. Then C is not well-covered since $\{x\}$ and $\{y, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it k. To prevent $\{yv, ys, yk\}$ and $\{yv, ym, yk\}$ from forming claws at y, we must have $s \sim k$ and $m \sim k$. Thus by planarity, k must be interior to the ysm-face. Now $d(y) = d(s) = d(m) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, q]$ containing y, so that $V(C) = \{x, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{x, k\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim q$.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, ws, wk\}$ and $\{wv, wn, wk\}$ from forming a claw at w, we must have $s \sim k$ and $n \sim k$. Thus by planarity, k must be interior to the wsn-face. Now $d(w) = d(s) = d(n) = 5$, and so by 3-connectivity, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m]$ containing w, so that $V(C) = \{v, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{v, k\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (as a subcase of $w \sim n$), and so $w \nsim n$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, ws, wm\}$ from forming a claw at w, we must have $s \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, m]$ containing v, so that $V(C) = \{u, v, y\}$.

Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Since $m \nsim q$ by birth, $m \nsim t$ by claw-freedom at t. Since $m \nsim n$ by birth, $m \nsim r$ by claw-freedom at r. Since m is not adjacent to either q or r, $m \nsim p$ by claw-freedom at p. Note that $m \nsim y$ by Claim 2.4/1.3.4.2. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it k. To prevent $\{yv, ys, yk\}$ from forming a claw at y, we must have $s \sim k$. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m]$ containing y, so that $V(C) = \{v, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{v, k\}$ are both maximal independent sets of C. Note that $m \nsim k$; otherwise we would have a forbidden subgraph centered at s, by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, k must be adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, m, j]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that j is not adjacent to m or t by birth. Recall that $m \nsim n$ by birth. Thus by Lemma 2.2, j must be adjacent to n. See Figure 68(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, j]$ containing x, so that $V(C) = \{x, y, z, t\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, y and z cannot grow, and $z \nsim y$. Thus by Lemma 2.3, if $\{w, p, j\}$ is an independent set, then G is not well-covered, a contradiction. Since $j \nsim m$ by birth, $j \nsim w$ by claw-freedom at w. Since $p \nsim s$ by birth, $p \nsim w$ by claw-freedom at w. Thus we must have $j \sim p$. To prevent $\{pr, pj, pq\}$ from forming a claw at p, we must have $j \sim q$ since $j \nsim r$ by birth. But then we have a forbidden subgraph centered at p by Claim 2.4/1.3.4.1. Hence k is not adjacent to an additional vertex and so w is not adjacent to an additional vertex (as a subcase of r adjacent to an additional vertex). Thus r is not adjacent to an additional vertex

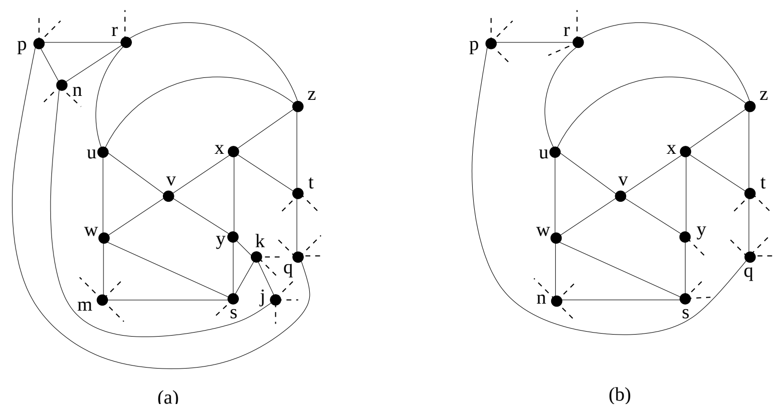


Figure 68: Proving that every vertex of degree four must lie on a K_4.

and so $d(r) = 3$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, ws, wn\}$ from forming a claw at w, we must have $s \sim n$. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(w) = 4$ and w cannot grow. See Figure 68(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, p]$ containing w, so that $V(C) = \{u, w, n\}$. Then C is not well-covered, since $\{w\}$ and $\{u, n\}$ are both maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Recall that $p \nsim t$ by birth. Since n is an additional vertex, and r is not adjacent to an additional vertex, $r \nsim n$. This together with the fact that $n \nsim q$ by birth, implies that $n \nsim p$ by claw-freedom at p. Since $n \nsim q$ by birth, $n \nsim t$ by claw-freedom at t. Note that $n \nsim y$ by birth. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, m]$ containing u, so that $V(C) = \{u, w, z, r\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that $m \nsim y$ by birth. Since m is adjacent to neither t nor p by birth, $m \nsim q$ by claw-freedom at q. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (as a subcase of $p \sim q$), and so $p \nsim q$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, ws, wn\}$ from forming a claw at w, we must have $s \sim n$. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(w) = 4$ and w cannot grow. See Figure 69(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y,t,p]$ containing w, so that $V(C) = \{u, w, n\}$. Then C is not well-covered, since $\{w\}$ and $\{u, n\}$ are both maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Recall that $p \nsim t$ by birth. Note that n is not adjacent to y or p by birth. Since $n \nsim q$ by birth, $n \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[x,p,q,m]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered, since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that $m \nsim p$ by birth. Also note that $m \nsim s$ by Claim 2.4/1.3.4.3, since n and s already share the vertex w, and w cannot grow. Thus by Lemma 2.2, either m is adjacent to q, or s is adjacent to an additional vertex.

Suppose $m \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[x,p,q]$ containing w, so that $V(C) = \{u, w, s, n\}$. Then C is not well-covered, since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to an additional vertex, or n is adjacent to an additional vertex. Suppose s is adjacent to an additional vertex; call it k. To prevent $\{sw, sy, sk\}$ from forming a claw at s, we must have $k \sim y$. Call this semi-known subgraph S. Let C be the component of $S - N[x,p,q,k]$ containing w, so that $V(C) = \{u, w, n\}$. Then

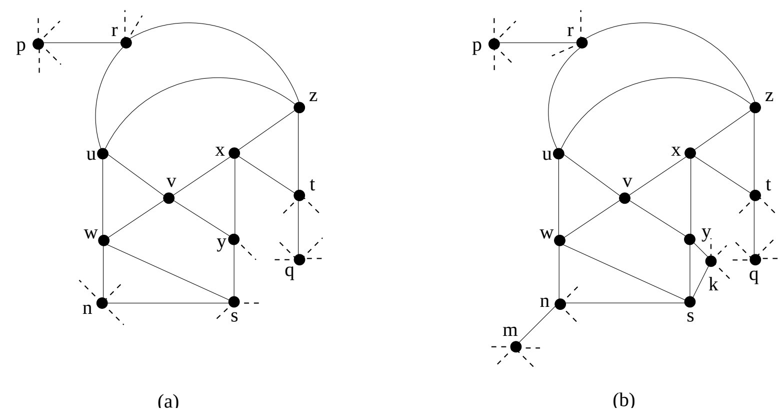


Figure 69: Proving that every vertex of degree four must lie on a K_4.

C is not well-covered, since $\{w\}$ and $\{u, n\}$ are both maximal independent sets of C. Note that k is not adjacent to x, p or q by birth. Also note that $k \nsim n$ by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it j. To prevent $\{nw, nm, nj\}$ from forming a claw at n, we must have $m \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, k, j]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that j is not adjacent to k or q by birth. Since neither j nor k is adjacent to p by birth, neither is adjacent to r by claw-freedom at r. Recall that $k \nsim q$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex and so $d(s) = 3$. Suppose n is adjacent to an additional vertex; call it k. To prevent $\{nw, nm, nk\}$ from forming a claw at n, we must have $m \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[x, q, p, k]$ containing w, so that $V(C) = \{u, w, s\}$. Then C is not well-covered, since $\{w\}$ and $\{u, s\}$ are both maximal independent sets of C. Note that k is not adjacent to x, q and p by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus n is not adjacent to an additional vertex and so $q \nsim m$.

Suppose s is adjacent to an additional vertex; call it k. To prevent $\{sw, sy, sk\}$

from forming a claw at s, we must have $y \sim k$. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. Thus $d(s) = 4$; otherwise an additional neighbor together with w and y would form a claw at s. See Figure 69(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, q, p, m]$ containing y, so that $V(C) = \{x, y, s, k\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and s cannot grow, and $x \nsim s$. Recall that $p \nsim q$ by a previous subcase (of the $w \sim s$ subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.5), $q \nsim m$ by the preceding subcase, and $m \nsim p$ by birth. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore $s \nsim w$.

Suppose $r \sim s$. Call the resulting semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that q is adjacent to neither w nor y by birth. Also recall that $r \nsim y$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.5). Note that $r \nsim w$; otherwise we would have a forbidden subgraph centered at u by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either r is adjacent to q, y is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Suppose $r \sim q$. To prevent $\{ru, rq, rs\}$ from forming a claw at r, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it p. See Figure 70(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r,p]$ containing x, so that $V(C) = \{v, x, y, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Since $p \nsim q$ by birth, $p \nsim r$ by claw-freedom at r. Note that $p \nsim y$ by birth, and $p \nsim t$ by planarity. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ from forming a claw at y, we must have $s \sim n$. Note that n could be in either the $uwvysr$-face or the $xysqt$-face. Also note that r does not have any additional neighbors in the $uwvysr$-face; otherwise the additional neighbor together with u and q would form a claw at r, contradicting the fact that G is claw-free. If n is in the $xysqt$-face, then y does not have any neighbors in the $uwvysr$-face; otherwise this neighbor together with n and v would be a claw at y, contradicting the fact that G is claw-free. But then $\{w, s\}$ is a 2-cut, separating p from the rest of the graph and contradicting the fact that G is claw-free. Thus n must be in the $uwvysr$-face. Call this semi-known subgraph S. Let C be the component of $S - N[r, p, n]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that n is not adjacent to p or r by birth. Also note that $n \nsim t$ by planarity. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it m. Now either m is in the $ztqr$-face or in the $xysqt$-face. Note that r does not have any additional neighbors in the $ztqr$-face; otherwise the additional neighbor together with u and s would form a claw at r, contradicting the fact that G is claw-free. Thus m is not in the $ztqr$-face; otherwise $\{t, q\}$ is a 2-cut, separating m from the rest of the graph and contradicting the fact that G is 3-connected. So m must be in the $xysqt$-face. Note that y does not have any additional neighbors in the $xysqt$-face; otherwise the additional neighbor together with v and n would form a claw at y, contradicting the fact that G is claw-free. Also s does not have any additional neighbors in the $xysqt$-face; otherwise the additional

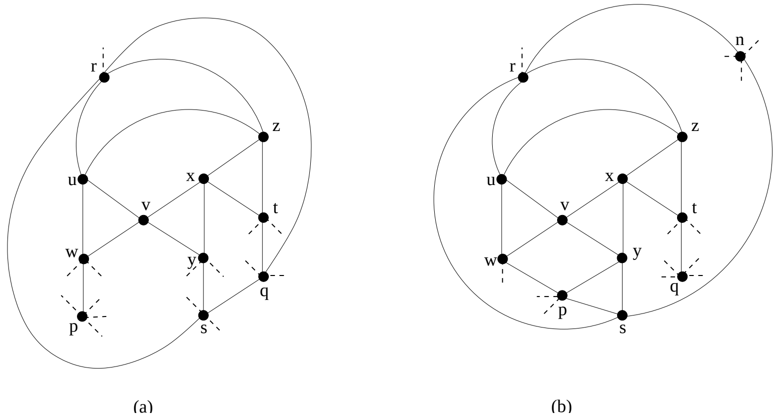


Figure 70: Proving that every vertex of degree four must lie on a K_4.

neighbor together with n and r would form a claw at s, contradicting the fact that G is claw-free. But then $\{t, q\}$ is a 2-cut separating m from the rest of the graph and contradicting the fact that G is 3-connected. Thus t is not adjacent to an additional vertex (as a subcase of y adjacent to an additional vertex), and so y is not adjacent to an additional vertex and hence $d(y) = 3$.

Suppose t is adjacent to an additional vertex; call it n. Now either n is in the $ztqr$-face or in the $xysqt$-face. Note that r does not have any additional neighbors in the $ztqr$-face; otherwise the additional neighbor together with u and s would form a claw at r, contradicting the fact that G is claw-free. Thus n is not in the $ztqr$-face; otherwise $\{t, q\}$ is a 2-cut, separating n from the rest of the graph and contradicting the fact that G is 3-connected. So n must be in the $xysqt$-face. Note that s does not have any additional neighbors in the $xysqt$-face; otherwise the additional neighbor together with y and r would form a claw at s, contradicting the fact that G is claw-free. But then $\{t, q\}$ is a 2-cut, separating n from the rest of the graph and contradicting the fact that G is 3-connected. Thus t is not adjacent to an additional vertex; and so $r \nsim q$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$

from forming a claw at y, we must have $s \sim p$. Note that p can be in either the $uwvysr$-face or the $xysrzt$-face. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that p is adjacent to neither q nor r by birth. Recall that $r \nsim q$ by the preceding subcase. Thus by Lemma 2.2, either p is adjacent to w, or w is adjacent to an additional vertex.

Suppose $p \sim w$. Then p must be in the $uwvysr$-face by planarity. Now w and y share the neighbor p. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that since $p \nsim q$ by birth, $p \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it n. To prevent $\{ru, rs, rn\}$ from forming a claw at r, we must have $n \sim s$. Suppose n is in the $uwpsr$-face. Then r does not have any additional neighbors in the $xysrzt$-face; otherwise the additional neighbor together with u and n would form a claw at r, contradicting the fact that G is claw-free. Also s does not have any additional neighbors in the $xysrzt$-face; otherwise the additional neighbor together with y and n would form a claw at s, contradicting the fact that G is claw-free. But then t is a cut-vertex, separating q from the rest of the graph and contradicting the fact that G is 3-connected. Hence, n must be in the $xysrzt$-face. See Figure 70(b) for an illustration. Note that r does not have any additional neighbors in the $uwpsr$-face; otherwise the additional neighbor together with u and n would form a claw at r, contradicting the fact that G is claw-free. Also s does not have any additional neighbors in the $uwpsr$-face; otherwise the additional neighbor together with y and n would form a claw at s, contradicting the fact that G is claw-free. Thus there can be no additional vertices in the $uwpsr$-face; otherwise $\{w, p\}$ would be a 2-cut, separating these additional vertices from the rest of the graph and contradicting the fact that G is claw-free. Since y cannot grow, there are also no additional vertices in the $vypw$-face by 3-connectivity. Hence w and p cannot grow and $d(w) = d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[x, n]$ containing w, so that $V(C) = \{u, w, p\}$. Then C is not well-covered since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $p \nsim w$.

Suppose w is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that n is not adjacent to r or p by birth, and $n \nsim t$ by planarity. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{ru, rs, rm\}$ from forming a claw at r, we must have $m \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing w, so that $V(C) = \{u, v, w, n\}$. Then C is not well-covered since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Recall that $s \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.3. Since $n \nsim p$ by birth, $n \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, either w is adjacent to an additional vertex or n is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wn, wk\}$ from forming a claw at w, we must have $n \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing w, so that $V(C) = \{u, v, w, n, k\}$. Then C is not well-covered since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that k is adjacent to neither s nor t by birth. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or one of n or k is adjacent to an additional vertex (the cases of n and k being adjacent to additional vertices are symmetric).

Suppose w is adjacent to an additional vertex; call it j. To prevent $\{wv, wn, wj\}$ and $\{wv, wk, wj\}$ from forming claws at w, we must have $n \sim j$ and $k \sim j$. Either j is interior to the wnk-face, k is interior to the wnj-face, or n is interior to the wkj-face. Since $d(w) = 5$, by Claim 2.4/1.2, whichever of j, k or n is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing w, so that $V(C) = \{u, v, w, n, k, j\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of n, k and j cannot grow, and v is not adjacent to any of n, k or j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus w is not adjacent to a fifth additional vertex and so $d(w) = 4$.

Without loss of generality, suppose n is adjacent to an additional vertex; call it j. See Figure 71(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, j]$ containing w, so that $V(C) = \{u, v, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that j is adjacent to neither s nor t by birth. Also note that $j \nsim k$ by Claim 2.4/1.3.4.3, since n and k already share the neighbor w, and w cannot grow. Thus by Lemma 2.2, k must be adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, m, j, i]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note $i \nsim j$ by birth, and neither i nor j is adjacent to q by planarity. Since neither i nor j is adjacent to s by each of their births, neither i nor j is adjacent to y by claw-freedom at y. Since $m \nsim p$ by birth, $m \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either q is adjacent to m, or i or j adjacent to m (the cases of i and j being adjacent to m are symmetric). Suppose $q \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, j, i]$ containing u, so that $V(C) = \{u, w, z, r\}$. Then every vertex of $C - u$ is adjacent

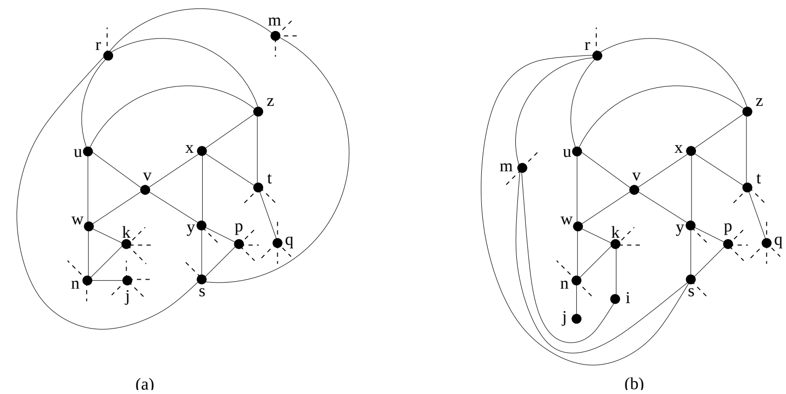


Figure 71: Proving that every vertex of degree four must lie on a K_4.

to u, vertices u, w and z cannot grow, and $w \nsim z$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim m$. Suppose, without loss of generality, that $i \sim m$. Then m must be in the $uwvysr$-face by planarity. See Figure 71(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, j, i]$ containing u, so that $V(C) = \{u, w, z, r\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, w and z cannot grow, and $w \nsim z$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence neither i nor j is adjacent to m, and so neither n nor k is adjacent to an additional vertex (as a subcase of w adjacent to an additional vertex). Thus w is not adjacent to an additional fourth neighbor, and so $d(w) = 3$.

Suppose n is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p, k]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Since $k \nsim s$ by birth, $k \nsim r$ by claw-freedom at r. Note that $k \nsim q$ by planarity. Thus by Lemma 2.2, we must have $k \sim p$. Thus p must be in the $uwvysr$-face by planarity. Suppose m is in the $uwvysr$-face as well. Then r does not have any additional neighbors in the $xysrzt$-face; otherwise the additional neighbor together with u and m would form a claw at r, contradicting the fact that

G is claw-free. Note that y does not have any additional neighbors in the $xysrzt$-face; otherwise the additional neighbor together with v and p would form a claw at y, contradicting the fact that G is claw-free. Also s does not have any additional neighbors in the $xysrzt$-face; otherwise the additional neighbor together with p and m would form a claw at s, contradicting the fact that G is claw-free. But then t is a cut-vertex, separating q from the rest of the graph and contradicting the fact that G is 3-connected. Hence m must be in the $xysrzt$-face. Now r does not have any additional neighbors in the $uwvysr$-face; otherwise the additional neighbor together with u and m would form a claw at r, contradicting the fact that G is claw-free. Note that y may now only have additional neighbors in the ysp-face, since by claw-freedom any additional neighbor must be adjacent to both p and s. Also s does not have any additional neighbors in the $uwvysr$-face; otherwise the additional neighbor together with p and m would form a claw at s, contradicting the fact that G is claw-free. But then $\{w, p\}$ is a 2-cut, separating n and k from the rest of the graph and contradicting the fact that G is 3-connected. Hence n is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore y is not adjacent to an additional vertex and so $d(y) = 3$.

Suppose w is adjacent to an additional vertex; call it p. See Figure 72(a) for an illustration. Note that by claw-freedom, each of r and s may have additional neighbors in at most one of the following two faces: the $uwvysr$-face, the $xysrzt$-face. By claw-freedom, they must share any additional neighbors they have. If they are not adjacent to an additional vertex in the $uwvysr$-face, then $\{w\}$ is a cut-vertex, separating p from the rest of the graph and contradicting the fact that G is 3-connected. If they are not adjacent to an additional vertex in the $xysrzt$-face, then $\{t\}$ is a cut-vertex, separating q from the rest of the graph thus contradicting the fact that G is 3-connected. Hence r and s must have additional neighbors in both faces. But this is a contradiction by claw-freedom. Hence w is not adjacent to an

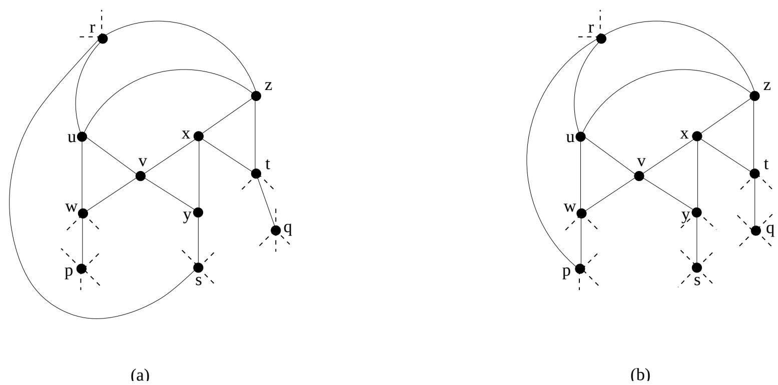


Figure 72: Proving that every vertex of degree four must lie on a K_4.

additional vertex and therefore $r \nsim s$.

Suppose w is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, p]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that p is adjacent to neither s nor t by birth. Thus by Lemma 2.2, either p is adjacent to r, or r is adjacent to an additional vertex.

Suppose $p \sim r$. See Figure 72(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, q]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that $r \nsim s$ and $s \nsim w$ by previous subcases (among the subcases of Claim 2.4/1.3.4.4.2.2.2.5), and $w \nsim q$ by birth. Thus by Lemma 2.2, either q is adjacent to r, q is adjacent to s, or w is adjacent to an additional vertex.

Suppose $q \sim r$. To prevent $\{ru, rp, rq\}$ from forming a claw at r, we must have $p \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Note that $p \nsim t$ by birth. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it n. To prevent $\{tx, tq, tn\}$ from forming a claw at t, we must have $n \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, n]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Since $n \nsim p$ by birth, n is adjacent to neither w or r by claw-freedom at each of w and r. Thus by Lemma 2.2, either n is adjacent to s, or w is adjacent to an additional vertex.

Suppose $n \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, n]$ containing v, so that $V(C) = \{v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex or w is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it m. To prevent $\{yv, ys, ym\}$ from forming a claw at y, we must have $s \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[r, n, m]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that m is adjacent to neither n nor r by birth. Thus by Lemma 2.2, either m is adjacent to w, or w is adjacent to an additional vertex.

Suppose $m \sim w$. To prevent $\{wv, wp, wm\}$ from forming a claw at w, we must have $p \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, m]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that since $m \sim w$, m must be in the $vwpqnsy$-face, and so by planarity $m \nsim t$. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it k. To prevent $\{ru, rp, rk\}$ and $\{ru, rq, rk\}$ from forming claws at r, we must have $p \sim k$ and $q \sim k$. Thus $d(p) = 5$, and so w cannot grow; otherwise an additional neighbor of w together with v and p would form a claw at w, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing u, so that $V(C) = \{u, w, z\}$.

Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim w$.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wp, wk\}$ from forming a claw, we must have $p \sim k$. See Figure 73(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m]$ containing w, so that $V(C) = \{v, w, k\}$. Then C is not well-covered since $\{w\}$ and $\{v, k\}$ are both maximal independent sets of C. Note that k is adjacent to neither r nor m by birth, and $k \nsim t$ by planarity. Recall that $m \nsim w$ from the preceding case. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or k is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it j. Either j is interior to the wpk-face, or k is interior to the wpj-face. Now $d(w) = d(p) = 5$ and so by Claim 2.4/1.2, whichever vertex of j or k is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, m]$ containing w, so that $V(C) = \{v, w, k, j\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of j and k cannot grow, and v is adjacent to neither k nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to a fifth additional vertex and so $d(w) = 3$. Suppose k is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, j]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Recall that $q \nsim y$ by birth. Since $j \nsim m$ by birth, $j \nsim y$ by claw-freedom at y. Since j is adjacent to neither t nor r by birth, $j \nsim q$ by claw-freedom at q. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence k is not adjacent to an additional vertex and so w is not adjacent to an additional vertex (as a subcase of y adjacent to an additional vertex). Therefore y is not adjacent to an additional vertex and so $d(y) = 3$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wp, wm\}$

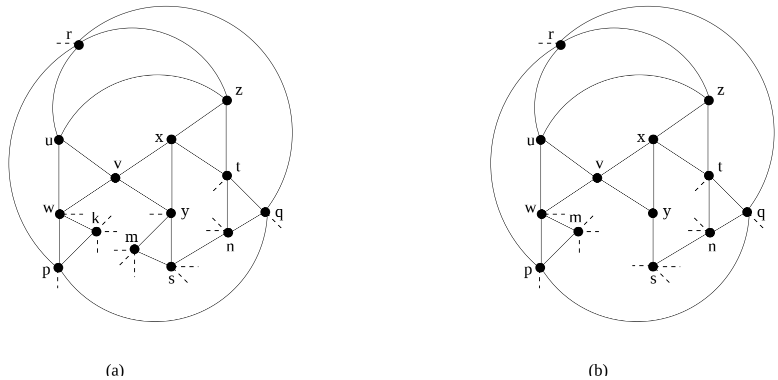


Figure 73: Proving that every vertex of degree four must lie on a K_4.

from forming a claw at w, we must have $m \sim p$. See Figure 73(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, m]$ containing x, so that $V(C) = \{v, x, t\}$. Then C is not well-covered since $\{x\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that $m \nsim r$ by birth. Since $m \nsim n$ by birth, $m \nsim s$ by claw-freedom at s. Note that $m \nsim t$ by planarity. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it k. To prevent $\{tx, tn, tk\}$ and $\{tx, tq, tk\}$ from forming claws at t, we must have $n \sim k$ and $q \sim k$. Thus by planarity k must be in the tqn-face. Now $d(t) = d(q) = 5$ and so by Claim 2.4/1.2, k cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, s, r]$ containing t, so that $V(C) = \{x, t, k\}$. Then C is not well-covered since $\{t\}$ and $\{x, k\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex (as a subcase of $n \sim s$), and so $n \nsim s$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wp, wm\}$ from forming a claw at w, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, m\}$. Then C is not well-covered since $\{w\}$ and $\{v, m\}$ are both maximal independent sets of C. Note that m is adjacent to neither r nor s by birth. Since $m \nsim n$ by birth, $m \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or m is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wp, wk\}$ and $\{wv, wm, wk\}$ from forming claws at w, we must have $p \sim k$ and $m \sim k$. Either k is interior to the wpm-face, or m is interior to the wpk-face. Now $d(w) = d(p) = 5$ and so by Claim 2.4/1.2, whichever vertex of k or m is interior cannot grow and has degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, m, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of m and k cannot grow, and v is adjacent to neither m nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus w is not adjacent to an additional fifth vertex. Suppose m is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, t]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since k is adjacent to neither r nor t by birth, $k \nsim q$ by claw-freedom at q. Since $k \nsim s$ by birth, $k \nsim y$ by claw-freedom at y. Note that since w cannot be adjacent to an additional vertex, C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore $q \nsim r$.

Suppose $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing v, so that $V(C) = \{v, w, x, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that $q \nsim r$ by the preceding case, and q is adjacent to neither w nor y by birth. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or w is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it n. To prevent $\{yv, ys, yn\}$ from forming a claw at y, we must have $s \sim n$. See Figure 74(a) for an illustration.

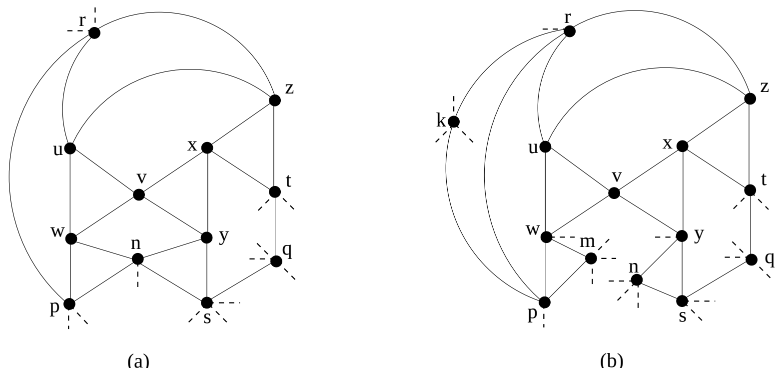


Figure 74: Proving that every vertex of degree four must lie on a K_4.

Call this semi-known subgraph S. Let C be the component of $S - N[r, q, n]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that n is adjacent to neither q nor r by birth. Thus by Lemma 2.2, either n is adjacent to w or w is adjacent to an additional vertex.

Suppose $n \sim w$. To prevent $\{wv, wp, wn\}$ from forming a claw at w, we must have $n \sim p$. Now w and y share the neighbor n. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ and $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$, $d(w) = 4$, and neither w nor y can grow. Call this semi-known subgraph S. Let C be the component of $S - N[t, n]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Since $n \nsim q$ by birth, $n \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{ru, rp, rm\}$ from forming a claw at r, we must have $p \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, m]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since $m \nsim n$ by birth, $m \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, we must have $m \sim q$. To prevent $\{qt, qs, qm\}$ from forming a claw at q, we must have $m \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[w, m]$ containing x, so that $V(C) = \{x, y, z, t\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, y and z cannot grow, and $y \nsim z$. Since $m \nsim n$ by birth, $m \nsim w$ by claw-freedom at w. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus $n \nsim w$.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wp, wm\}$ from forming a claw at w, we must have $p \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, m]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered, since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that m is adjacent to neither r nor n by birth. Since $m \nsim q$ by birth, $m \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it k. To prevent $\{ru, rp, rk\}$ from forming a claw at r, we must have $p \sim k$. See Figure 74(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, m\}$. Then C is not well-covered, since $\{w\}$ and $\{v, m\}$ are both maximal independent sets of C. Since m is adjacent to neither n nor q by birth, $m \nsim s$ by claw-freedom at s. Recall that $s \nsim w$ by a previous subcase (of the t adjacent to an additional vertex subcase; a subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.5). Thus by Lemma 2.2, either w is adjacent to an additional vertex, or m is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it j. To prevent $\{wv, wp, wj\}$ and $\{wv, wm, wj\}$ from forming claws at w, we must have $p \sim j$ and $m \sim j$. Either j is interior to the wpm-face or m is interior to the wpj-face. Now $d(w) = d(p) = 5$ and so by Claim 2.4/1.2, whichever vertex of j or m is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, m, j\}$. Then every vertex of $C - w$

is adjacent to w, vertices w, v and one of m or j cannot grow, and v is adjacent to neither m nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional fifth neighbor.

Suppose m is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, n, j]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $j \nsim r$ by birth. Since j is adjacent to neither s nor t by birth, $j \nsim q$ by claw-freedom at q. Thus by Lemma 2.2, we must have $n \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, j]$ containing v, so that $V(C) = \{v, w, x, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, w and x cannot grow, and $w \nsim x$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex, and therefore w is not adjacent to an additional vertex (as a subcase of y adjacent to an additional vertex). Therefore y is not adjacent to an additional vertex and so $d(y) = 3$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wp, wn\}$ from forming a claw at w, we must have $p \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, n]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that since $n \nsim q$ by birth, n is adjacent to neither s nor t by claw-freedom at those vertices. Also note that $n \nsim r$ by birth. Thus by Lemma 2.2, vertex s must be adjacent to an additional vertex; call it m. To prevent $\{sy, sq, sm\}$ from forming a claw at s, we must have $m \sim q$. See Figure 75(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, m]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $m \nsim t$ by birth. Since m is adjacent to neither n nor r by birth, $m \nsim p$ by claw-freedom at p. Thus by Lemma 1.4, G is not

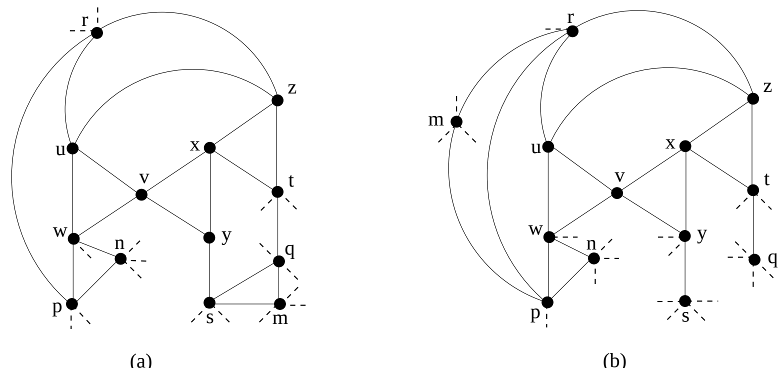


Figure 75: Proving that every vertex of degree four must lie on a K_4.

well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $q \nsim s$.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wp, wn\}$ from forming a claw at w, we must have $p \sim n$. Note that n could be interior to the $uwpr$-face, or n could be in the exterior face. Call this semi-known subgraph S. Let C be the component of $S - N[w, s, q]$ containing z, so that $V(C) = \{x, z, r\}$. Then C is not well-covered, since $\{z\}$ and $\{x, r\}$ are both maximal independent sets of C. Recall that $q \nsim s$ (by the preceding subcase) and $s \nsim w$ (by a previous subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.5), and $q \nsim w$ by birth. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{ru, rp, rm\}$ from forming a claw at r, we must have $m \sim p$. One of the two possible cases is shown in Figure 75(b). Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, n\}$. Then C is not well-covered, since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that n is adjacent to neither r nor s by birth. Since $n \nsim q$ by birth, $n \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either w is adjacent to an additional vertex or n is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wp, wk\}$

and $\{wv, wn, wk\}$ from forming claws at w, we must have $p \sim k$ and $n \sim k$. Either k is interior to the wpn-face, or n is interior to the wpk-face. Now $d(w) = d(p) = 5$ and so by Claim 2.4/1.2, whichever of k or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing w, so that $V(C) = \{v, w, n, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of k and n cannot grow, and v is adjacent to neither k nor n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional fifth neighbor.

Suppose n is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, q, k]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that k is adjacent to neither r nor s by birth, and C cannot grow. Thus by Lemma 2.2, we must have $k \sim q$. Note that this forces n and k to lie in the exterior face (not the $uwpr$-face). Call this semi-known subgraph S. Let C be the component of $S - N[y, p, k]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $k \nsim s$ by birth, $k \nsim y$ by claw-freedom at y. Note that $k \nsim p$ by Claim 2.4/1.3.4.3, since p and n share the neighbor w and w cannot grow. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it j. To prevent $\{tx, tq, tj\}$ from forming a claw at t, we must have $q \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m, j, k]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that j is adjacent to neither k nor y by birth. Recall that w is not adjacent to an additional vertex by the preceding case. Thus by Lemma 2.2, either m is adjacent to k, or m is adjacent to j.

Suppose $m \sim k$. To prevent $\{kn, kq, km\}$ from forming a claw at k, we must have

$m \sim n$ (since neither m nor n is adjacent to q by their respective births). But then we have a forbidden subgraph centered at m by Claim 2.4/1.3.4.1. Thus $m \nsim k$.

Suppose $m \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, k, j]$ containing w, so that $V(C) = \{v, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{v, p\}$ are both maximal independent sets of C. Note that $j \nsim s$ by planarity, and $j \nsim p$ by birth. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it i. To prevent $\{pw, pr, pi\}$ and $\{pw, pm, pi\}$ from forming claws at p, we must have $r \sim i$ and $m \sim i$. Thus i must be in the prm-face by planarity. Now $d(p) = d(r) = 5$ and so by Claim 2.4/1.2, vertex i cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, z, k, j]$ containing p, so that $V(C) = \{w, p, i\}$. Then C is not well-covered, since $\{p\}$ and $\{w, i\}$ are both maximal independent sets of C. Since $k \nsim s$ by birth, $k \nsim y$. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim j$, and so n is not adjacent to an additional vertex. Therefore w is not adjacent to an additional vertex and so $p \nsim r$.

Suppose r is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing w, so that $V(C) = \{u, w, p\}$. Then C is not well-covered since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Note that $n \nsim t$ by birth. Since $n \nsim p$ by birth, $n \sim w$ by claw-freedom at w. Since $n \nsim s$ by birth, $n \nsim y$ by claw-freedom at y. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Recall that $p \nsim t$ by birth. Also recall that $t \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.1. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or p is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wp, wm\}$ from forming a claw at w, we must have $p \sim m$. See Figure 76(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing

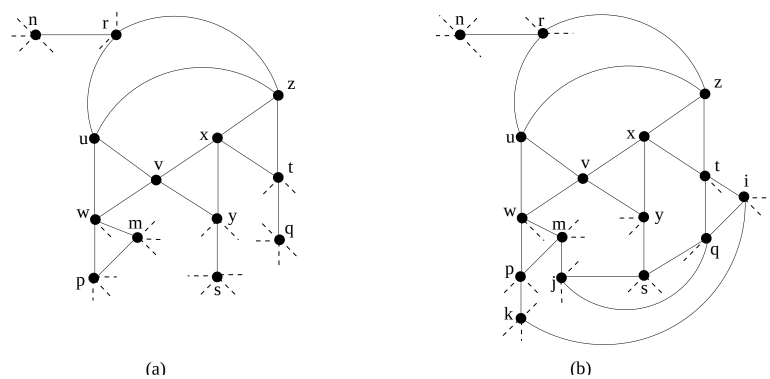


Figure 76: Proving that every vertex of degree four must lie on a K_4.

w, so that $V(C) = \{u, w, p, m\}$. Then C is not well-covered since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, p is adjacent to an additional vertex, or m is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wp, wk\}$ and $\{wv, wm, wk\}$ from forming claws at w, we must have $p \sim k$ and $m \sim k$. Either k is interior to the wpm-face, m is interior to the wpk-face, or p is interior to the wmk-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever vertex of k, m or p is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing w, so that $V(C) = \{u, w, p, m, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u, and one of p, m and k cannot grow, and u is adjacent to none of p, m or k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to a fifth additional vertex and so $d(w) = 4$.

Suppose p is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n, k]$ containing w, so that $V(C) = \{u, w, m\}$. Then C is not well-covered since $\{w\}$ and $\{u, m\}$ are both maximal independent sets of C. Note that k is not adjacent to y, t or n by birth.

Also note that $k \nsim m$ by Claim 2.4/1.3.4.3, since m and p already share the neighbor w, and $w \nsim k$. Thus by Lemma 2.2, m must be adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n, k, j]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that j is not adjacent to n, k or y by birth, and that C cannot grow. Thus by Lemma 2.2, either n is adjacent to q, or one of j and k is adjacent to q (the arguments for $j \sim q$ and $k \sim q$ are symmetric).

Suppose $n \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, k, j]$ containing u, so that $V(C) = \{u, w, z, r\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, w and z cannot grow, and $w \nsim z$. Note that $j \nsim q$; otherwise $\{qt, qn, qj\}$ would be a claw at q since j is adjacent to neither t nor n by birth. Also note that $k \nsim q$; otherwise $\{qt, qn, qk\}$ would be a claw at q, since k is adjacent to neither n nor t by birth. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $n \nsim q$.

Suppose, without loss of generality, that $j \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n, j]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $j \nsim k$ by birth, $j \nsim p$ by claw-freedom at p. Recall $p \nsim n$ by birth. Thus by Lemma 2.2, vertex t must be adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, k, j, i]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that i is not adjacent to j, n or y by birth. Thus by Lemma 2.2, we must have $k \sim i$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing y, so that $V(C) = \{v, x, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Since

p is adjacent to neither t or j by birth of p and j respectively, $p \nsim q$ by claw-freedom at q. Recall that $p \nsim r$ by a previous subcase (of the w adjacent to an additional vertex subcase, which is one of the subcases of the t adjacent to an additional vertex subcase; a subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.5). Since $n \nsim q$ by birth, $q \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either q is adjacent to s, y is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose $q \sim s$. To prevent $\{qt, qs, qj\}$ from forming a claw at q, we must have $s \sim j$. See Figure 76(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, q, n]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{y, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it h. To prevent $\{yv, ys, yh\}$ from forming a claw at y, we must have $s \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, q]$ containing y, so that $V(C) = \{x, y, h\}$. Then C is not well-covered since $\{y\}$ and $\{x, h\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or h is adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it g. To prevent $\{yv, ys, yg\}$ and $\{yv, yh, yg\}$ from forming claws at y, we must have $g \sim s$ and $g \sim h$. Either g is interior to the ysh-face, or h is interior to the ysg face. Now $d(y) = d(s) = 5$ and so by Claim 2.4/1.2, whichever vertex of g or h is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, q, r]$ containing y, so that $V(C) = \{x, y, h, g\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and one of h or g cannot grow, and x is adjacent to neither h nor g. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $d(y) = 4$. Suppose h is adjacent to an additional vertex; call it g. Call this semi-known subgraph S. Let C be the component of $S - N[w, q, n, g]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered since $\{x\}$ and

$\{y, z\}$ are both maximal independent sets of C. Note that g is not adjacent to w or q by birth, and $g \nsim n$ by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence h is not adjacent to an additional vertex and so $q \nsim s$.

Suppose y is adjacent to an additional vertex; call it h. To prevent $\{yv, ys, yh\}$ from forming a claw at y, we must have $s \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing y, so that $V(C) = \{v, x, y, s, h\}$. Then C is not well-covered since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that h is not adjacent to p, q or r by birth, and $q \nsim s$ by the preceding case. Thus by Lemma 2.2, either y is adjacent to an additional vertex, s is adjacent to an additional vertex, or h is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it g. To prevent $\{yv, ys, yg\}$ and $\{yv, yh, yg\}$ from forming claws at y, we must have $s \sim g$ and $h \sim g$. Either g is interior to the ysh-face, h is interior to the ysg-face, or s is interior to the yhg-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, whichever vertex of g, h or s is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing y, so that $V(C) = \{v, x, y, s, h, g\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and one of s, h and g cannot grow, and x is adjacent to none of s, h and g. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional fifth neighbor, and so $d(y) = 4$.

Suppose s is adjacent to an additional vertex; call it g. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p, g]$ containing y, so that $V(C) = \{v, x, y, h\}$. Then C is not well-covered, since $\{y\}$ and $\{v, h\}$ are both maximal independent sets of C. Note that g is not adjacent to r, q or p by birth. Also note that $h \nsim g$ by Claim 2.4/1.3.4.3, since h and s share y as a neighbor and y cannot grow. Thus by Lemma 2.2, h must be adjacent to an additional vertex; call it

f. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n, g, f]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that f is not adjacent to p or g by birth. Since neither f nor g is adjacent to q by birth, neither f nor g is adjacent to t by claw-freedom at t. Also note that neither f nor g is adjacent to n by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex (as a subcase of y adjacent to an additional vertex).

Suppose h is adjacent to an additional vertex; call it g. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p, g]$ containing y, so that $V(C) = \{v, x, y, s\}$. Then C is not well-covered, since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that g is not adjacent to p, q or r by birth, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus h is not adjacent to an additional vertex and so y is not adjacent to an additional fourth vertex. Thus $d(y) = 3$.

Suppose s is adjacent to an additional vertex; call it h. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n, h]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $h \nsim p$ by birth. Since $h \nsim q$ by birth, $h \nsim t$ by claw-freedom at t. Also note that $h \nsim n$ by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex and so neither j nor k is adjacent to q. Therefore p is not adjacent to an additional vertex.

Suppose m is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n, k]$ containing w, so that $V(C) = \{u, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Note that k is not adjacent to y, t or n by birth. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Thus $d(w) = 3$.

Suppose p is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n, m]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that m is adjacent to neither n nor y by birth. Recall that $q \nsim y$ by birth. Thus by Lemma 2.2, either m is adjacent to q, or n is adjacent to q.

Suppose $m \sim q$. See Figure 77(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, n]$ containing w, so that $V(C) = \{v, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{v, p\}$ are both maximal independent sets of C. Note that since n is adjacent to neither t nor m by birth, and $m \nsim t$ by birth, $n \nsim q$ by claw-freedom at q. Thus by Lemma 2.2, either q is adjacent to s, q is adjacent to p, or p is adjacent to an additional vertex.

Suppose $q \sim s$. To prevent $\{qt, qm, qs\}$ from forming a claw at q, we must have $m \sim s$, since $s \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.3. Call this semi-known subgraph S. Let C be the component of $S - N[z, m]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $m \nsim y$ by birth. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it k. To prevent $\{yv, ys, yk\}$ from forming a claw at y, we must have $s \sim k$. Note that k could be in either the $vysmpw$-face, or the $xysqt$-face. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, q]$ containing y, so that $V(C) = \{x, y, k\}$. Then C is not well-covered, since $\{y\}$ and $\{x, k\}$ are both maximal independent sets of C. Since r is adjacent to neither s nor t, $r \nsim q$ by claw-freedom at q. Note that $k \nsim r$ by planarity. Thus by Lemma 2.2, either q is

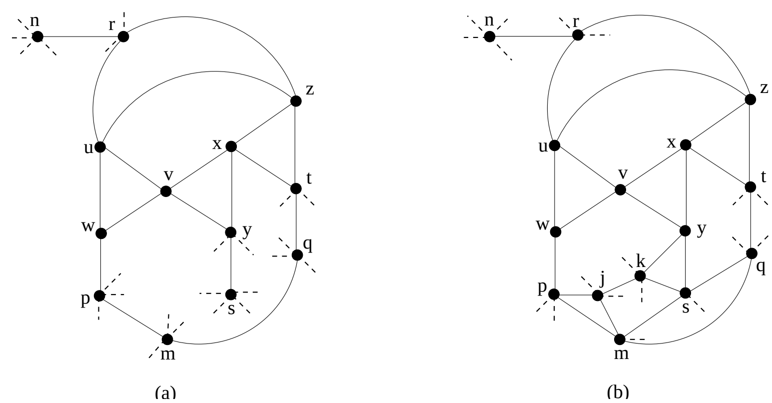


Figure 77: Proving that every vertex of degree four must lie on a K_4.

adjacent to k, y is adjacent to an additional vertex, or k is adjacent to an additional vertex.

Suppose $q \sim k$. Note that this forces k to be in the $xysqt$-face. To prevent $\{qt, qm, qk\}$ from forming a claw at q, we must have $k \sim t$, since $k \nsim m$ by birth. But then we have a forbidden subgraph centered at k, by Claim 2.4/1.3.4.1. Thus $q \nsim k$.

Suppose y is adjacent to an additional vertex; call it j. To prevent $\{yv, ys, yj\}$ and $\{yv, yk, yj\}$ from forming claws at y, we must have $s \sim j$ and $k \sim j$. Either j is interior to the ysk-face, or k is interior to the ysj-face. Now $d(y) = d(s) = 5$, and so by Claim 2.4/1.2, whichever vertex of j or k is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, q]$ containing y, so that $V(C) = \{x, y, k, j\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and one of k or j cannot grow, and x is adjacent to neither k nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $d(y) = 4$.

Suppose k is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[z, m, j]$ containing v, so that

$V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $j \sim m$. Note that $j \nsim s$ by Claim 2.4/1.3.4.3, since k and s share the neighbor y, and y cannot grow. Thus to prevent $\{mp, ms, mj\}$ from forming a claw at m, we must have $j \sim p$. See Figure 77(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, q, j]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that $j \nsim q$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus k is not adjacent to an additional vertex and so $q \nsim s$.

Suppose $q \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, q]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Since $p \nsim r$ by a previous subcase (of the w adjacent to an additional vertex subcase of the t adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.5), and $r \nsim t$ otherwise we would have a forbidden subgraph centered at z by Claim 2.4/1.3.4.1, $r \nsim q$ by claw-freedom at q. (Recall that $t \nsim p$ by birth.) Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim p$.

Suppose p is adjacent to an additional vertex; call it k. To prevent $\{pw, pm, pk\}$ from forming a claw at p, we must have $m \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n, k]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that k is adjacent to neither q nor n by birth. Since $k \nsim s$ by birth, $k \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence p is not adjacent to an additional vertex, and therefore $m \nsim q$.

Suppose $n \sim q$. See Figure 78(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, m]$ containing u, so that $V(C) =$

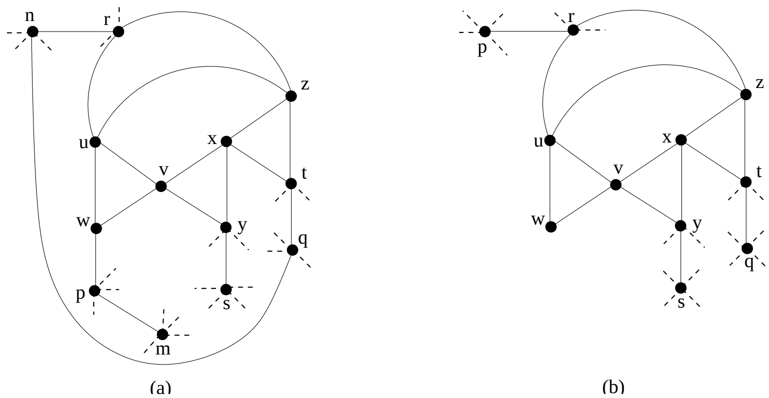


Figure 78: Proving that every vertex of degree four must lie on a K_4.

$\{u, w, z, r\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, w and z cannot grow, and $w \nsim z$. Recall that $m \nsim y$ by birth, and $m \nsim q$ by the preceding subcase. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $n \nsim q$, and so p is not adjacent to an additional vertex. Therefore r is not adjacent to an additional vertex (as a subcase of w adjacent to an additional vertex), and so w is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it p. See Figure 78(b) for an illustration. Note that w is adjacent to neither y nor r; otherwise there would be forbidden subgraphs centered at t and u, respectively, by Claim 2.4/1.3.4.1. Recall that $w \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.1. Also note that $q \nsim w$ by birth, and $s \nsim w$ by a previous subcase (of the t adjacent to an additional vertex subcase; a subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.5). But then since w is not adjacent to an additional vertex (and this includes p), $d(w) = 2$, contradicting the fact that G is 3-connected. Hence r is not adjacent to an additional vertex (as a subcase of t adjacent to an additional vertex), and so t is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it q. Note that t is adjacent to neither y nor r; otherwise there would be forbidden subgraphs centered at x and z, respectively, by Claim 2.4/1.3.4.1. Recall that $w \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.1, and $t \nsim s$ by Claim 2.4/1.3.4.4.2.2.2.3. But then since t is not adjacent to an additional vertex (and this includes q) by the preceding case, $d(t) = 2$, contradicting the fact that G is 3-connected. Hence r is not adjacent to an additional vertex and so z is not adjacent to an additional vertex. Therefore $d(z) = 3$ and we have proved Claim 2.4/1.3.4.4.2.2.2.5. ∎

Note that since $d(z) = 3$, $d(u) = 3$ as well. Recall that $u \nsim t$ by Claim 2.4/1.3.4.4.2.2.1. If u was adjacent to an additional vertex, then this vertex together with v and z would form a claw at u, contradicting the fact that G is claw-free.

Claim 2.4/1.3.4.4.2.2.2.6: The vertex t is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.2.2.6: Suppose, by way of contradiction, that t is adjacent to an additional vertex; call it r. Since G is 3-connected, $\{u, v\}$ is not a 2-cut and so there must be a path from w to the set $\{y, t, s, r\}$ that does not pass through either u or v. Recall that $w \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.1. Note that $w \nsim y$ otherwise there would be a forbidden subgraph centered at v by Claim 2.4/1.3.4.1. Thus either w is adjacent to s, w is adjacent to r, or w is adjacent to an additional vertex.

Suppose $w \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered, since $\{x\}$ and $\{z, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to r, y is adjacent to r, or y is adjacent to an additional vertex.

Suppose $w \sim r$. Now w and y share the neighbor s. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(w) = 4$ and w cannot grow. To prevent

$\{wv, ws, wr\}$ from forming a claw at w, we must have $r \sim s$. See Figure 79(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w]$ containing x, so that $V(C) = \{x, y, z, t\}$. Then C is not well-covered, since $\{x\}$ and $\{z, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[u, r]$ containing y, so that $V(C) = \{x, y, q\}$. Then C is not well-covered, since $\{x\}$ and $\{y, q\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either r is adjacent to y, r is adjacent to q, y is adjacent to an additional vertex, or q is adjacent to an additional vertex. Suppose $r \sim y$. But then $\{rw, ry, rt\}$ is a claw at r, contradicting the fact that G is claw-free. Hence $r \nsim y$. Suppose $r \sim q$. To prevent $\{rt, rw, rq\}$ from forming a claw at r, we must have $q \sim t$. (Recall that $q \nsim w$ by birth.) But then we have a forbidden subgraph centered at q, by Claim 2.4/1.3.4.1. Hence $r \nsim q$. Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $s \sim p$ and $q \sim p$. Either p is interior to the ysq-face, or q is interior to the ysp-face. Now $d(y) = d(s) = 5$, and so by Claim 2.4/1.2, whichever vertex of p and q is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[u, r]$ containing y, so that $V(C) = \{x, y, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x, and one of q and p cannot grow, and x is adjacent to neither q nor p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $d(y) = 4$. Suppose q is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing x, so that $V(C) = \{x, y, z, t\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, y, and z cannot grow, and $y \nsim z$. Thus by Lemma 2.3, G is not

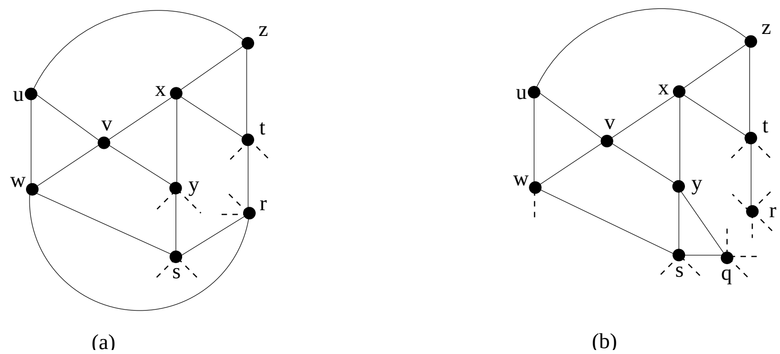


Figure 79: Proving that every vertex of degree four must lie on a K_4.

well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so y is not adjacent to an additional vertex. Therefore $d(y) = 3$.

Suppose t is adjacent to an additional vertex; call it q. To prevent $\{tx, tr, tq\}$ from forming a claw at t we must have $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[u, q]$ containing y, so that $V(C) = \{x, y, s\}$. Then C is not well-covered, since $\{y\}$ and $\{x, s\}$ are both maximal independent sets of C. Note that $q \nsim s$; otherwise $\{sw, sy, sq\}$ would be a claw at s since $q \nsim w$ by birth. Recall y is not adjacent to an additional vertex, by the preceding case. Note that s is not adjacent to an additional vertex; otherwise this additional vertex together with w and y would form a claw at s, contradicting the fact that G is claw-free. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex and so $w \nsim r$.

Suppose $r \sim y$. Note w and y share the neighbor s. Since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. To prevent $\{yv, ys, yr\}$ from forming a claw at y, we must have $r \sim s$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(j). Since G is not a graph from

Figure 2, this semi-known subgraph of G must grow. Since there are no possible additional edges between known vertices, w, s, t or r must be adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it q. Note w and y share the neighbor s. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(w) = 4$ and w cannot grow. To prevent $\{wv, ws, wq\}$ from forming a claw at w, we must have $q \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[t, q]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $q \sim t$. To prevent $\{tx, tr, tq\}$ from forming a claw at t, we must have $q \sim r$. But then we have a forbidden subgraph centered at s, by Claim 2.4/1.3.4.1. Hence w is not adjacent to an additional vertex, and $d(w) = 3$. Suppose s is adjacent to an additional vertex; call it q. But then $\{sw, sy, sq\}$ is a claw at s, contradicting the fact that G is claw-free. Hence s is not adjacent to an additional vertex and $d(s) = 3$. Now since neither w nor s is adjacent to an additional vertex, t and r cannot be adjacent to an additional vertex either; otherwise $\{t, r\}$ would be a 2-cut, separating any additional vertices from the rest of the graph and contradicting the fact that G is 3-connected. Thus $r \nsim y$.

Suppose y is adjacent to an additional vertex; call it q. Note w and y share the neighbor s. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(y) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $s \sim q$. See Figure 79(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Recall that $w \nsim r$ by a previous subcase (of the $w \sim s$ subcase; the first subcase of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it p. Note w and y share the neighbor s. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u share a neighbor and are not adjacent, we may conclude by symmetry that $d(w) < 5$ when y and w share a neighbor and are not adjacent. Thus here we may say that $d(w) = 4$ and w cannot grow. To prevent $\{wv, ws, wp\}$ from forming a claw at w, we must have $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing y, so that $V(C) = \{v, y, q\}$. Then C is not well-covered, since $\{y\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that $p \nsim q$ otherwise we would have a forbidden subgraph centered at s, by Claim 2.4/1.3.4.1. Also note that neither p nor q is adjacent to r by their births. Thus by Lemma 2.2, q must be adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[w, r, n]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered, since $\{x\}$ and $\{y, z\}$ are both maximal independent sets of C. Note that $n \nsim r$ by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $w \nsim s$.

Suppose $w \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s\}$. Then C is not well-covered, since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $r \nsim y$; otherwise $\{ry, rw, rt\}$ would be a claw at r, since $t \nsim w$ by Claim 2.4/1.3.4.4.2.2.2.1. Also note that $r \nsim s$; otherwise $\{rw, rt, rs\}$ would form a claw at r, since all possible adjacencies between s, t and w have been eliminated by previous cases. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $s \sim q$. See Figure 80(a) for an illustration.

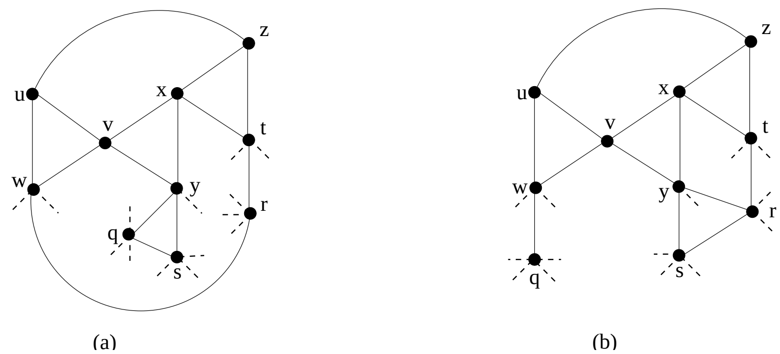


Figure 80: Proving that every vertex of degree four must lie on a K_4.

Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s, q\}$. Then C is not well-covered, since $\{y\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or either s or q is adjacent to an additional vertex (the case of s adjacent to an additional vertex is symmetric with the case of q adjacent to an additional vertex, so we need only consider one of these cases).

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $s \sim p$ and $q \sim p$. Either p is interior to the ysq-face, q is interior to the ysp-face, or s is interior to the yqp-face. Now $d(y) = 5$, and so by Claim 2.4/1.2, whichever of vertices of p, q or s is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing y, so that $V(C) = \{v, y, s, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and one of s, q and p cannot grow, and v is adjacent to none of s, q, or p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to a fifth additional neighbor, and so $d(y) = 4$.

Suppose, without loss of generality, that s is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing y, so that $V(C) = \{v, y, q\}$. Then C is not well-covered, since $\{y\}$ and

$\{v, q\}$ are both maximal independent sets of C. Note that p is not adjacent to r or z by birth. Also note that $p \nsim q$ by Claim 2.4/1.3.4.3, since s and q share the neighbor y and y cannot grow. Thus by Lemma 2.2, q must be adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Note that $n \nsim p$ by birth. Since $n \nsim r$ by birth, $n \nsim t$ by claw-freedom at t. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence neither s nor q is adjacent to an additional vertex (as a subcase of y adjacent to an additional vertex), and so y is not adjacent to an additional vertex. Thus $d(y) = 3$.

Suppose s is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[t, q]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Since $q \nsim r$ by birth, $q \nsim t$ by claw-freedom at t. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence s is not adjacent to an additional vertex and so $r \nsim w$.

Suppose w is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[w, s]$ containing t, so that $V(C) = \{x, z, t, r\}$. Then C is not well-covered since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Note that s and r are not adjacent to w by the two preceding subcases of Claim 2.4/1.3.4.4.2.2.2.6. Recall that $t \nsim w$ and $s \nsim t$ by Claim 2.4/1.3.4.4.2.2.2.1 and Claim 2.4/1.3.4.4.2.2.2.3, respectively. Thus by Lemma 2.2, either r is adjacent to s, t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose $r \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[w, r]$ containing x, so that $V(C) = \{x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{y, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either

r is adjacent to y, or y is adjacent to an additional vertex.

Suppose $r \sim y$. Note t and y share the neighbor r. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u (on opposite sides of the bow-tie centered at v) share a neighbor and are not adjacent, we may conclude, since v was an arbitrarily chosen vertex of G with degree four, that $d(y) < 5$ when y and t (on opposite sides of the bow-tie centered at x) share a neighbor and are not adjacent. Thus here we may say that $d(y) = 4$ and y cannot grow. See Figure 80(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y,t]$ containing w, so that $V(C) = \{u, w, q\}$. Then C is not well-covered since $\{w\}$ and $\{u, q\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either t is adjacent to q, w is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose $t \sim q$. To prevent $\{tx, tr, tq\}$ from forming a claw at t, we must have $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[u, q]$ containing y, so that $V(C) = \{x, y, s\}$. Then C is not well-covered since $\{y\}$ and $\{x, s\}$ are both maximal independent sets of C. Note that $q \nsim s$ otherwise we would have a forbidden subgraph centered at r, by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, s must be adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, p]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Note that $p \nsim t$ by planarity. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim t$.

Suppose w is adjacent to an additional vertex; call it p. To prevent $\{wv, wq, wp\}$ from forming a claw at w, we must have $q \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, t]$ containing w, so that $V(C) = \{u, w, q, p\}$. Then C is not well-covered since $\{w\}$ and $\{u, q\}$ are both maximal independent sets of C. Note that $p \nsim t$ by birth, and $q \nsim t$ by the preceding subcase. Thus by Lemma

2.2, either w is adjacent to an additional vertex, q is adjacent to an additional vertex, or p is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wq, wn\}$ and $\{wv, wp, wn\}$ from forming claws at w, we must have $q \sim n$ and $p \sim n$. Either n is interior to the wqp-face, p is interior to the wqn-face, or q is interior to the wpn-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, whichever vertex of q, p, or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, t]$ containing w, so that $V(C) = \{u, w, q, p, n\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u and one of q, p and n cannot grow, and u is adjacent to none of q, p or n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to a fifth additional vertex and so $d(w) = 4$.

Suppose q is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing w, so that $V(C) = \{u, w, p\}$. Then C is not well-covered, since $\{w\}$ and $\{u, p\}$ are both maximal independent sets of C. Note that $n \nsim t$ by birth. Also note that $p \nsim n$ by Claim 2.4/1.3.4.3, since q and p already share the neighbor w, and w cannot grow. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it k. To prevent $\{tx, tr, tk\}$ from forming a claw at t, we must have $r \sim k$. Note that since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u (on opposite sides of the bow-tie centered at v) share a neighbor and are not adjacent, we may conclude, since v was an arbitrarily chosen vertex of G with degree four, that $d(t) < 5$ when y and t (on opposite sides of the bow-tie centered at x) share a neighbor and are not adjacent. Thus here we may say that $d(t) = 4$ and t cannot grow. See Figure 81(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, m, k]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{z, w\}$ are both maximal independent sets of C. Note that $m \nsim n$ by birth. Thus by Lemma 2.2, either k is adjacent to m, or k is adjacent to n. Suppose $k \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, m]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $m \nsim q$ by Claim 2.4/1.3.4.3, since q and p already share the neighbor w, and w cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus $k \nsim m$. Suppose $k \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus $k \nsim n$, and so q is not adjacent to an additional vertex.

Suppose p is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing w, so that $V(C) = \{u, w, q\}$. Then C is not well-covered, since $\{w\}$ and $\{u, q\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus p is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore $d(w) = 3$.

Suppose q is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it n. To prevent $\{tx, tr, tn\}$ from forming a claw at t, we must have $r \sim n$. Note that

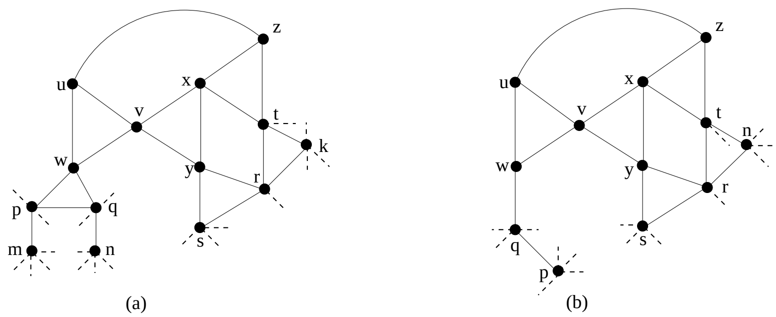


Figure 81: Proving that every vertex of degree four must lie on a K_4.

since by Claim 2.4/1.3.4.4.2.1, $d(x) < 5$ when x and u (on opposite sides of the bow-tie centered at v) share a neighbor and are not adjacent, we may conclude, since v was an arbitrarily chosen vertex of G with degree four, that $d(t) < 5$ when y and t (on opposite sides of the bow-tie centered at x) share a neighbor and are not adjacent. Thus here we may say that $d(t) = 4$ and t cannot grow. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, we must have $n \sim p$. See Figure 81(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[v, s, q]$ containing t, so that $V(C) = \{z, t, n\}$. Then C is not well-covered, since $\{t\}$ and $\{z, n\}$ are both maximal independent sets of C. Note $n \nsim q$ by birth, and $n \nsim s$ otherwise we would have a forbidden subgraph centered at r, by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either q is adjacent to s, or n is adjacent to an additional vertex.

Suppose $q \sim s$. To prevent $\{qw, qs, qp\}$ from forming a claw at q, we must have $s \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[v, s]$ containing t, so that $V(C) = \{z, t, n\}$. Then C is not well-covered, since $\{t\}$ and $\{z, n\}$ are both maximal independent sets of C. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it m. To prevent $\{nt, np, nm\}$ from forming a claw at n, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[q, y, m]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $m \nsim s$ by birth, $m \nsim q$ by claw-freedom at q. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim s$.

Suppose n is adjacent to an additional vertex; call it m. To prevent $\{nt, np, nm\}$ from forming a claw at n, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[q, y, m]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $m \nsim s$ by birth, $m \nsim q$ by claw-freedom at q. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so q is not adjacent to an additional vertex. Therefore $r \nsim y$.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, ys, yp\}$ from forming a claw at y, we must have $p \sim s$. Note that p could be in either the $xysrt$-face or the exterior face. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing t, so that $V(C) = \{x, z, t, r\}$. Then C is not well-covered since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Note that p is adjacent to neither w nor r by birth. Since $p \nsim r$, we have $p \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it n. To prevent $\{tx, tr, tn\}$ from forming a claw at t, we must have $r \sim n$. Note that n could be in either the $xysrt$-face or the exterior face. One of the two possible cases is shown in Figure 82(a). Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing t, so that $V(C) = \{x, z, t, r, n\}$. Then C is not well-covered since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Note that n is adjacent to neither w

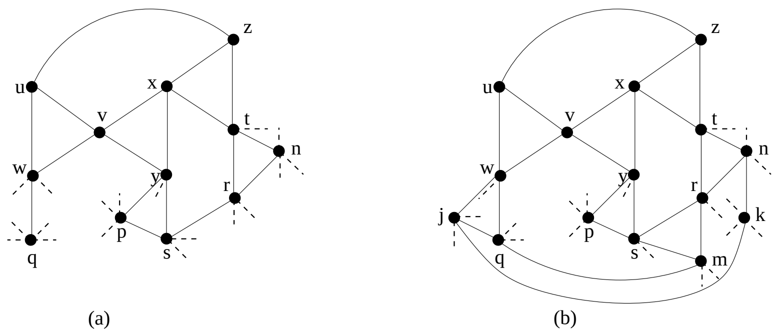


Figure 82: Proving that every vertex of degree four must lie on a K_4.

nor p by birth. Thus by Lemma 2.2, either t is adjacent to an additional vertex, r is adjacent to an additional vertex, or n is adjacent to an additional vertex. Suppose t is adjacent to an additional vertex; call it m. To prevent $\{tx, tr, tm\}$ and $\{tx, tn, tm\}$ from forming claws at t, we must have both $r \sim m$ and $n \sim m$. Either m is interior to the trn-face, or n is interior to the trm-face. Now $d(t) = 5$ and so by Claim 2.4/1.2, whichever vertex of m or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, p]$ containing t, so that $V(C) = \{x, z, t, r, n, m\}$. Then every vertex of $C - t$ is adjacent to t, vertices t, x and one of n and m cannot grow, and x is adjacent to neither n nor m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence t is not adjacent to an additional fifth vertex.

Suppose r is adjacent to an additional vertex; call it m. To prevent $\{rt, rs, rm\}$ from forming a claw at r, we must have $s \sim m$. Note that m could be in either the $xysrt$-face or the exterior face. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, m]$ containing t, so that $V(C) = \{x, z, t, n\}$. Then C is not well-covered since $\{t\}$ and $\{x, n\}$ are both maximal independent sets of C. Note that m is adjacent to neither w nor p by birth. Also note that $m \nsim n$; otherwise we would have a forbidden subgraph centered at r by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it k. Call this semi-known subgraph S.

Let C be the component of $S - N[y, q, m, k]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Recall that t is not adjacent to an additional vertex, by the preceding case. Then by Lemma 2.2, either q is adjacent to y, q is adjacent to m, q is adjacent to k, or q is adjacent to t.

Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ and $\{yv, yp, yq\}$ from forming claws at y, we must have $s \sim q$ and $p \sim q$. Now $d(y) = d(s) = 5$ and so by Claim 2.4/1.2, p cannot grow and hence must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[u, r]$ containing y, so that $V(C) = \{x, y, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and p cannot grow, and $x \nsim p$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim y$.

Suppose $q \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, m]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since $n \nsim p$ by birth, $n \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it j. To prevent $\{wv, wq, wj\}$ from forming a claw at w, we must have $q \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m, k, j]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that j is adjacent to neither m nor y by birth. Thus by Lemma 2.2, we must have $j \sim k$. See Figure 82(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, k]$ containing w, so that $V(C) = \{u, v, w, q\}$. Then C is not well-covered since $\{w\}$ and $\{u, q\}$ are both maximal independent sets of C. Recall that k is not adjacent to w by birth. Since k is adjacent to neither w nor m by birth, $k \nsim q$ by claw-freedom at q. Thus by Lemma 2.2, either q is adjacent to s, w is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose $q \sim s$. To prevent $\{sy, sr, sq\}$ from forming a claw at s, we must have $r \sim q$. Note that all of s, r and q have degree five. Thus by claw-freedom, none of y, t and w, because of their respective relationships to s, r and q, can grow. But then no matter whether p is interior to the $vwqmsy$-face or to the $xysrt$-face, $\{y, s\}$ is a 2-cut, separating p from the rest of the graph and contradicting the fact that G is 3-connected. Hence $q \nsim s$.

Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ and $\{wv, wj, wi\}$ from forming claws at w, we must have $q \sim i$ and $j \sim i$. Thus by planarity, i must be in the wqj-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, i cannot grow and $d(i) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, k]$ containing w, so that $V(C) = \{u, v, w, q, i\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u and i cannot grow, and $u \nsim i$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex. Suppose q is adjacent to an additional vertex; call it i. To prevent $\{qw, qm, qi\}$ from forming a claw at q, we must have $i \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, k, i]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that $i \nsim k$ by birth. Since i is adjacent to neither s nor t by birth, i is adjacent to neither y nor r by claw-freedom at each vertex. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so $q \nsim m$.

Suppose $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, k]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since w is adjacent to neither t (by Claim 2.4/1.3.4.4.2.2.2.1) nor s (by a previous subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.6), $w \nsim r$ by claw-freedom at r. Thus by Lemma

2.2, w must be adjacent to an additional vertex; call it j. To prevent $\{wv, wq, wj\}$ from forming a claw at w, we must have $q \sim j$. Note that j may be in either the $vwqknrsy$-face or the $uwqkntz$-face. See Figure 83(a) for a possible illustration. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, k]$ containing w, so that $V(C) = \{u, w, j\}$. Then C is not well-covered since $\{w\}$ and $\{u, j\}$ are both maximal independent sets of C. Note that $j \nsim k$ by birth. Since j is adjacent to neither r nor y by birth, $j \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or j is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wq, wi\}$ and $\{wv, wj, wi\}$ from forming claws at w, we must have $q \sim i$ and $j \sim i$. Either i is interior to the wqj-face, or j is interior to the wqi-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, whichever vertex of i or j is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, k]$ containing w, so that $V(C) = \{u, w, j, i\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u and one of j and i cannot grow, and u is adjacent to neither j nor i. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex, and $d(w) = 4$. Suppose j is adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, k, i]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that $i \nsim k$ by birth. Recall $t \nsim i$ since t is not adjacent to an additional vertex. Since $i \nsim s$ by birth, i is adjacent to neither y nor r by claw-freedom at those vertices. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence j is not adjacent to an additional vertex and so $q \nsim k$.

Suppose $q \sim t$. To prevent $\{tx, tr, tq\}$ and $\{tx, tn, tq\}$ from forming claws at t, we must have $r \sim q$ and $n \sim q$. Then $d(t) = d(r) = 5$, and k is interior to either the rqn-face or the to tqn-face. But then $\{q, n\}$ is a 2-cut, separating k from the rest of

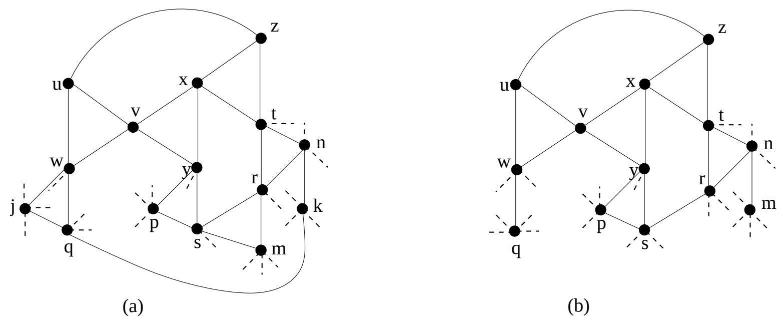


Figure 83: Proving that every vertex of degree four must lie on a K_4.

the graph, and contradicting the fact that G is 3-connected. Hence $q \nsim t$, and so r is not adjacent to an additional vertex (as a subcase of t adjacent to an additional vertex).

Suppose n is adjacent to an additional vertex; call it m. See Figure 83(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, p, m]$ containing t, so that $V(C) = \{x, z, t, r\}$. Then C is not well-covered, since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Note that m is adjacent to neither w nor p by birth. Also note that C cannot grow since t and r cannot be adjacent to additional vertices. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus n is not adjacent to an additional vertex and so t is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it n. To prevent $\{rt, rs, rn\}$ from forming a claw at r, we must have $s \sim n$, since $t \nsim s$ by Claim 2.4/1.3.4.4.2.2.2.3 and t is not adjacent to additional vertices by the preceding case. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $n \nsim p$ by birth, $n \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either n is adjacent to q, y is adjacent to q, or t is adjacent to q.

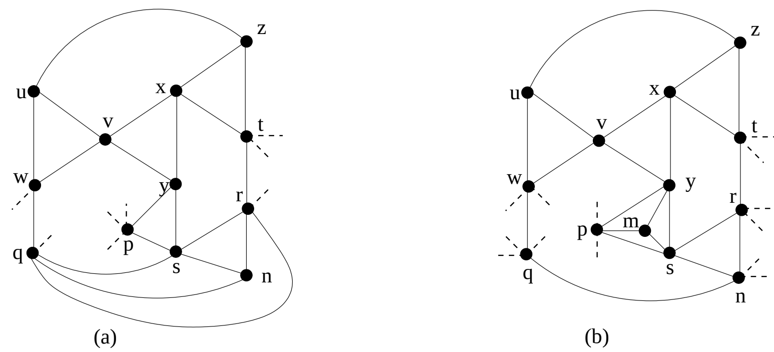


Figure 84: Proving that every vertex of degree four must lie on a K_4.

Suppose $n \sim q$. See Figure 84(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, r, q]$ containing y, so that $V(C) = \{x, y, p\}$. Then C is not well-covered, since $\{y\}$ and $\{x, p\}$ are both maximal independent sets of C. Then by Lemma 2.2, either q is adjacent to r, q is adjacent to y, q is adjacent to p, y is adjacent to an additional vertex, or p is adjacent to an additional vertex.

Suppose $q \sim r$. To prevent $\{rt, rs, rq\}$ from forming a claw at r, either q is adjacent to t, or q is adjacent to s. Suppose $q \sim t$. But then we have a forbidden subgraph centered at r, by Claim 2.4/1.3.4.1. Thus $q \nsim t$. Suppose $q \sim s$. Now $d(s) = 5$ and so by Claim 2.4/1.2, n cannot grow and so $d(n) = 3$. Suppose $q \sim y$. Then we must have $q \sim p$ otherwise $\{yv, yq, yp\}$ would form a claw at y. But then $d(q) = 6$, contradicting Claim 2.4/1.1. Hence $q \nsim y$. Thus $d(y) = 4$; otherwise an additional neighbor together with v and s would form a claw at y. Call this semi-known subgraph S. Let C be the component of $S - N[w, t]$ containing s, so that $V(C) = \{y, s, p, n\}$. Then every vertex of $C - s$ is adjacent to s, vertices s, y and n cannot grow, and $y \nsim n$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim r$.

Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ and $\{yv, yp, yq\}$ from forming claws at y, we must have $s \sim q$ and $p \sim q$. But then we have the forbidden subgraph shown in Figure 5(b) and centered at s. Thus $q \nsim y$.

Suppose $q \sim p$. But then $\{qw, qp, qn\}$ is a claw at w, since n and p are not adjacent to w by their respective births, and $n \nsim p$ otherwise we would have a forbidden subgraph centered at s by Claim 2.4/1.3.4.1. Thus $q \nsim p$.

Suppose y is adjacent to an additional vertex; call it m. To prevent $\{yv, ys, ym\}$ and $\{yv, yp, ym\}$ from forming claws at y, we must have $s \sim m$ and $p \sim m$. Either m is interior to the ysp-face, or p is interior to the ysm-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, whichever vertex of m or p is interior cannot grow and must have degree three. One of the two possible cases is shown in Figure 84(b). Call this semi-known subgraph S. Let C be the component of $S - N[u, r, q]$ containing y, so that $V(C) = \{x, y, p, m\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and one of p and m cannot grow, and x is adjacent to neither p nor m. Recall $q \nsim r$ by a previous subcase (among the $n \sim q$ subcases of the r adjacent to an additional vertex subcase of the y adjacent to an additional vertex subcase of the $r \sim s$ subcase of the w adjacent to an additional vertex subcase; a subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex.

Suppose p is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[z, m, n]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Since m is adjacent to neither r nor q by birth, $m \nsim n$ by claw-freedom at n. Recall that $n \nsim w$ by birth. Since $m \nsim q$ by birth, $m \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $q \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, m, k]$

containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Since $m \nsim r$ by birth, $m \nsim t$ by claw-freedom at t. Note that k is not adjacent to m or n by birth. Note that $n \nsim t$ by Claim 2.4/1.3.4.3, since r and n already share the neighbor s, and $s \nsim t$. Thus by Lemma 2.2, we must have $k \sim t$. To prevent $\{tx, tr, tk\}$ from forming a claw at t, we must have $k \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, m]$ containing v, so that $V(C) = \{u, v, w, y\}$. Then every vertex of $C - v$ is adjacent to v, vertices v, u and y cannot grow, and $u \nsim y$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence p is not adjacent to an additional vertex and so $n \nsim q$.

Suppose $y \sim q$. To prevent $\{yv, ys, yq\}$ and $\{yv, yp, yq\}$ from forming claws at y, we must have $s \sim q$ and $p \sim q$. Now $d(y) = d(s) = 5$ and so by Claim 2.4/1.2, p cannot grow and $d(p) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[u, r]$ containing y, so that $V(C) = \{x, y, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, x and p cannot grow, and $x \nsim p$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $q \nsim y$.

Suppose $q \sim t$. To prevent $\{tx, tr, tq\}$ from forming a claw at t, we must have $r \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of G. Recall that $w \nsim r$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it m. To prevent $\{wv, wq, wm\}$ from forming a claw at w, we must have $q \sim m$. Either m is in the $uwqtz$-face, or m is in the $vwqrsy$-face. One of the two possible cases is shown in Figure 85(a). Call this semi-known subgraph S. Let C be the component of $S - N[y, n, m]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both

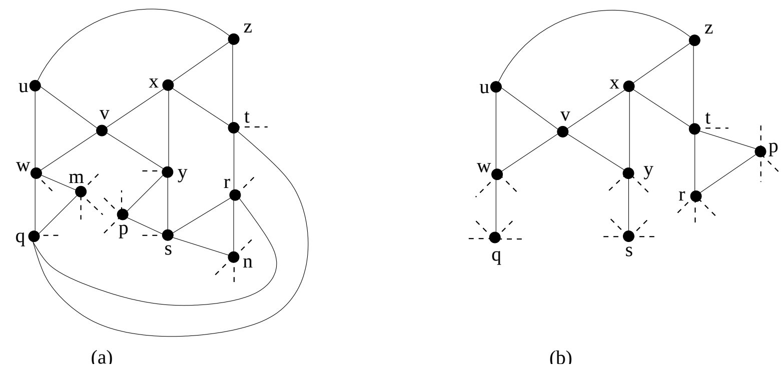


Figure 85: Proving that every vertex of degree four must lie on a K_4.

maximal independent sets of G. Note that $m \nsim y$ by birth. Since $m \nsim r$ by birth, $m \nsim t$ by claw-freedom at t. Note that t is not adjacent to an additional vertex, by a previous subcase (among the subcases of the y adjacent to an additional vertex subcase of the $r \sim s$ subcase of the w adjacent to an additional vertex subcase; a subcase among the subcases of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 2.2, m must be adjacent to n. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing w, so that $V(C) = \{u, v, w, m\}$. Then C is not well-covered, since $\{w\}$ and $\{u, m\}$ are both maximal independent sets of G. Note that since m is adjacent to neither y nor r by birth, $m \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or m is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wq, wk\}$ and $\{wv, wm, wk\}$ from forming claws at w, we must have $q \sim k$ and $m \sim k$. Thus by planarity, k is interior to the wqm-face. Now $d(w) = d(q) = 5$ and so by Claim 2.4/1.2, k cannot grow and $d(k) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing w, so that $V(C) = \{u, v, w, m.k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, u and k cannot grow, and $u \nsim k$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex.

Suppose m is adjacent to an additional vertex; call it k. To prevent $\{mw, mn, mk\}$ from forming a claw at m, we must have $n \sim k$ since $n \nsim w$ by birth, and w is not adjacent to an additional vertex (i.e. k) by the preceding case. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, k]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since k is adjacent to neither t nor s by birth, k is adjacent to neither y nor r by claw-freedom at each. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex and so $q \nsim t$. Therefore r is not adjacent to an additional vertex and so y is not adjacent to an additional vertex. Thus $r \nsim s$.

Suppose t is adjacent to an additional vertex; call it p. To prevent $\{tx, tr, tp\}$ from forming a claw at t, we must have $r \sim p$. See Figure 85(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, s]$ containing t, so that $V(C) = \{x, z, t, r, p\}$. Then C is not well-covered since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Recall $s \nsim w$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 2.2, either t is adjacent to an additional vertex, or either r or p is adjacent to an additional vertex (the arguments for r adjacent to an additional vertex, and p adjacent to an additional vertex are symmetric).

Suppose t is adjacent to an additional vertex; call it n. To prevent $\{tx, tr, tn\}$ and $\{tx, tp, tn\}$ from forming claws at t, we must have $r \sim n$ and $p \sim n$. Either n is interior to the trp-face, p is interior to the trn-face, or r is interior to the tpn-face. Now $d(t) = 5$, and so by Claim 2.4/1.2, whichever of these three vertices is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[w, s]$ containing t, so that $V(C) = \{x, z, t, r, p, n\}$. Then every vertex of $C-t$ is adjacent to t, vertices t, x and one of r, p and n cannot grow, and x is adjacent to none of r, p or n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence t is not adjacent to a fifth additional vertex.

Suppose, without loss of generality, that r is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[w, s, n]$ containing t, so that $V(C) = \{x, z, t, p\}$. Then C is not well-covered since $\{t\}$ and $\{x, p\}$ are both maximal independent sets of C. Note that n is adjacent to neither s nor w by birth. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n, m]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $m \nsim n$ by birth. Since neither m nor n is adjacent to s by birth, neither is adjacent to y by claw-freedom at y. Thus by Lemma 2.2, either q is adjacent to y, q is adjacent to m or n (these two arguments are symmetric), or q is adjacent to t.

Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $s \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since y is adjacent to neither t or n, $y \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $q \sim k$. See Figure 86(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, m, k]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $k \nsim y$ by birth. Recall that t is not adjacent to an additional vertex by the preceding case. Thus by Lemma 2.2, either k is adjacent to m, or k is adjacent to n.

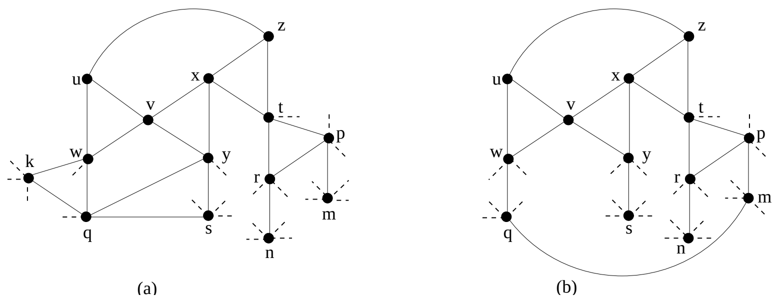


Figure 86: Proving that every vertex of degree four must lie on a K_4.

Suppose $k \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that $m \nsim r$ by Claim 2.4/1.3.4.3, since r and p share the neighbor t, and $t \nsim m$ since t is not adjacent to additional vertices. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it j. To prevent $\{wv, wq, wj\}$ and $\{wv, wk, wj\}$ from forming claws at w, we must have $q \sim j$ and $k \sim j$. Thus by planarity, j must be interior to the wqk-face. Now $d(w) = d(q) = 5$, and so by Claim 2.4/1.2, j cannot grow and $d(j) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, m]$ containing w, so that $V(C) = \{u, w, j\}$. Then C is not well-covered since $\{w\}$ and $\{u, j\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $k \nsim m$.

Suppose $k \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Recall that neither p nor n are adjacent to w by birth. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it j. To prevent $\{wv, wq, wj\}$ and $\{wv, wk, wj\}$ from forming claws at w, we must have $q \sim j$ and $k \sim j$. Thus by planarity, j must be interior to the wqk-face. Now

$d(w) = 5$ and so by Claim 2.4/1.2, j cannot grow and $d(j) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, n]$ containing w, so that $V(C) = \{u, w, j\}$. Then C is not well-covered since $\{w\}$ and $\{u, j\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $k \nsim n$ and so $q \nsim y$. Suppose, without loss of generality, that $q \sim m$. See Figure 86(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, r, m]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Recall that $m \nsim w$ by birth. Since $m \nsim s$ by birth, $m \nsim y$ by claw-freedom at y. Since $r \nsim s$ by a previous subcase (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.6), $r \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $q \sim k$. Note that k could be either in the $uwqmptz$-face or the $vwqmptxy$-face. Call this semi-known subgraph S. Let C be the component of $S - N[y, n, m, k]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that k is adjacent to neither y nor m by birth. Recall that t is not adjacent to an additional vertex (which includes n, m, and k, since we concluded that t could not be adjacent to an additional vertex *before* n, m, and k were born). Thus by Lemma 2.2, we must have $k \sim n$. Note that this forces k to lie interior to the $vwqmptxy$-face. Call this semi-known subgraph S. Let C be the component of $S - N[u, s, m, k]$ containing t, so that $V(C) = \{x, t, r\}$. Then C is not well-covered since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Since $s \nsim w$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.6), and n is adjacent to neither w nor s by birth, $s \nsim k$ by claw-freedom at k. Note that $k \nsim r$ by birth. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it j. To prevent $\{rt, rn, rj\}$ from forming a claw at r, we must have $n \sim j$. Call this semi-known subgraph S. Let C

be the component of $S - N[y, m, k, j]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that j is adjacent to neither k nor m by birth. Since $k \nsim s$ by birth, $k \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is adjacent to neither m nor n.

Suppose $q \sim t$. To prevent $\{tx, tr, tq\}$ and $\{tx, tp, tq\}$ from forming claws at t, we must have $q \sim r$ and $q \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Recall that since $r \nsim s$ by a previous subcase (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.6), $r \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it k. To prevent $\{wv, wq, wk\}$ from forming a claw at w, we must have $q \sim k$. Now $d(q) = d(t) = 5$ and so q and t cannot grow. Note that m is either in the tqp-face or in the rqp-face. If m is in the tqp-face, then p is a cut-vertex, separating m from the rest of the graph, and contradicting the fact that G is 3-connected. Hence m is not in the tqp-face. If m is in the rqp-face, then $\{r, p\}$ is a 2-cut, separating m (and possibly n) from the rest of the graph, and contradicting the fact that G is 3-connected. Thus m is not in the rqp-face. Hence $t \nsim q$, and so neither r nor p is adjacent to an additional vertex (as a subcase of t adjacent to an additional vertex). Therefore t is not adjacent to an additional vertex. Thus either t is adjacent to q or $d(t) = 3$.

Suppose r is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, p]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Since $p \nsim s$ by birth, $p \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either q is adjacent to y, q is adjacent to p, or q is adjacent to t.

Suppose $q \sim y$. To prevent $\{yv, ys, yq\}$ from forming a claw at y, we must have $q \sim s$. See Figure 87(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Recall that $r \nsim w$ by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.6). Since $r \nsim s$ by a previous subcase (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.6), $r \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wv, wq, wn\}$ from forming a claw at w, we must have $n \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $n \nsim y$ by birth. Thus by Lemma 2.2, we must have $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[v, s, n]$ containing z, so that $V(C) = \{z, t, r\}$. Then C is not well-covered, since $\{t\}$ and $\{z, r\}$ are both maximal independent sets of C. Note that $n \nsim s$; otherwise we would have a forbidden subgraph centered at q by Claim 2.4/1.3.4.1. Also note that $n \nsim r$ by birth. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{rt, rp, rm\}$ from forming a claw at r, we must have $p \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[v, m, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $m \nsim n$ by birth. Since $m \nsim s$ by birth, $m \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus $q \nsim y$.

Suppose $q \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[v, s, q]$ containing t, so that $V(C) = \{z, t, r\}$. Then C is not well-covered, since $\{t\}$ and $\{z, r\}$ are both maximal independent sets of C. Since s is not adjacent to w by a previous subcase (among the subcases of Claim 2.4/1.3.4.4.2.2.2.6), and

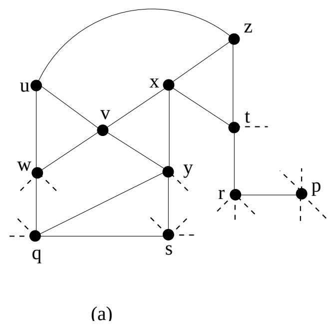


Figure 87: Proving that every vertex of degree four must lie on a K_4.

p, by the birth of p, $s \nsim q$ by claw-freedom at q. Thus by Lemma 2.2, either q is adjacent to t, q is adjacent to r, or r is adjacent to an additional vertex. Suppose $q \sim t$. Then $d(t) = 4$ and t cannot grow by a previous subcase (of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.6). But then $\{qw, qt, qp\}$ is a claw at q, since $p \nsim w$ by birth. Thus $q \nsim t$ and $d(t) = 3$. Suppose $q \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Recall $q \nsim y$ by the preceding subcase (of the r adjacent to an additional vertex subcase of the w adjacent to an additional vertex subcase of Claim 2.4/1.3.4.4.2.2.2.6). Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim r$. Suppose r is adjacent to an additional vertex; call it n. To prevent $\{rt, rp, rn\}$ from forming a claw at r, we must have $p \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $n \nsim q$ by birth. Since $n \nsim s$ by birth, $n \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $q \nsim p$.

Suppose $q \sim t$. Then $d(t) = 4$ and t cannot grow. To prevent $\{tx, tr, tq\}$ from forming a claw at t, we must have $r \sim q$. See Figure 87(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, r]$ containing u, so that $V(C) = \{u, w, z\}$. Then C is not well-covered, since $\{u\}$ and $\{w, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, w is adjacent to an additional vertex; call it n. To prevent $\{wv, wq, wn\}$ from forming a claw at w, we must have $q \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, n]$ containing u, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $n \nsim y$ by birth. Thus by Lemma 2.2, we must have $n \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[u, s, n]$ containing t, so that $V(C) = \{x, t, r\}$. Then C is not well-covered, since $\{t\}$ and $\{x, r\}$ are both maximal independent sets of C. Since s is not adjacent to w, by a previous subcase (among the subcase of Claim 2.4/1.3.4.4.2.2.2.6), and is not adjacent to p, by the birth of p, $s \nsim n$ by claw-freedom at n. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{rt, rp, rm\}$ from forming a claw at r, we must have $p \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[y, m, n]$ containing z, so that $V(C) = \{u, z, t\}$. Then C is not well-covered, since $\{z\}$ and $\{u, t\}$ are both maximal independent sets of C. Note that $m \nsim n$ by birth. Since $m \nsim s$ by birth, $m \nsim y$ by claw-freedom at y. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $q \nsim t$, and so r is not adjacent to an additional vertex.

Therefore w is not adjacent to an additional vertex and so t is not adjacent to an additional vertex. Thus we have proved Claim 2.4/1.3.4.4.2.2.2.6. ■

Thus y is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.4.2.2.2. ■

Claim 2.4/1.3.4.4.2.2.3: The vertex t is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.2.2.3: Suppose, by way of contradiction, that t is adjacent to an additional vertex; call it s. See Figure 88(a) for an illustration. Now y is

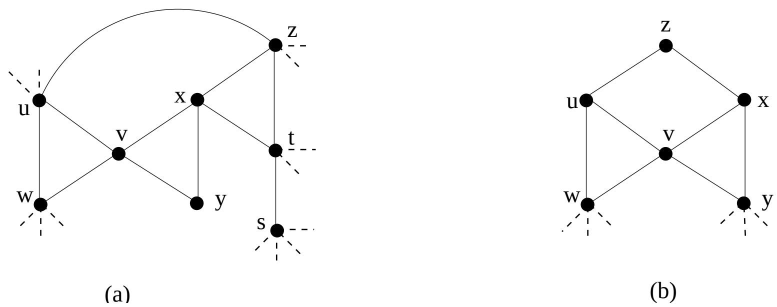


Figure 88: Proving that every vertex of degree four must lie on a K_4.

not adjacent to u, w, z or t; otherwise any one of these adjacencies would form a forbidden subgraph centered at either v or x by Claim 2.4/1.3.4.1 or Claim 2.4/1.3.4.2. Since y is not adjacent to any additional vertices by Claim 2.4/1.3.4.4.2.2.2, $y \nsim s$. But then $d(y) = 2$, contradicting the fact that G is 3-connected. Hence t is not adjacent to an additional vertex. ■

Therefore $d(x) \neq 4$, and we have proved Claim 2.4/1.3.4.4.2.2. ■

Claim 2.4/1.3.4.4.2.1 together with Claim 2.4/1.3.4.4.2.2 imply that we must have $d(x) = 3$. Note that by similar arguments, we may now assume that $d(u) = 3$ as well. Now we will show that this too is not possible.

Claim 2.4/1.3.4.4.2.3: The vertex x does not have degree three.

Proof of Claim 2.4/1.3.4.4.2.3: By way of contradiction, suppose $d(x) = 3$. See Figure 88(b) for an illustration. Recall that by Claim 2.4/1.3.4.4.1, $z \nsim y$, and by a symmetric argument, we may assume $z \nsim w$. Thus z cannot be adjacent to an additional vertex; otherwise this additional neighbor together with x and u would form a claw at z. But then $d(z) = 2$ contradicting the fact that G is 3-connected. Hence $d(x) \neq 3$. ■

Claim 2.4/1.3.4.4.2.1 shows that $d(x) < 5$, and Claims 2.4/1.3.4.4.2.2 and 2.4/1.3.4.4.2.3 show that x cannot have degree four or three. But $d(x) > 2$ since

G is 3-connected. Thus we have a contradiction, and so we have proved Claim 2.4/1.3.4.4.2, and $z \nsim u$. ■

Claim 2.4/1.3.4.4.3: The vertex z is not adjacent to w.

Proof of Claim 2.4/1.3.4.4.3: By way of contradiction, suppose $z \sim w$. We claim that both w and x must be adjacent to additional vertices. Suppose that w is not adjacent to an additional vertex (i.e. $d(w) = 3$). Since G is 3-connected, $\{v, x\}$ is not a 2-cut and so there must be a path from y to z that does not pass through either v or x. Since $z \nsim y$ by Claim 2.4/1.3.4.4.1, z must be adjacent to an additional vertex in the $vwzxy$-face; call it t. To prevent $\{zw, zx, zt\}$ from forming a claw at z, we must have $x \sim t$. Then x cannot be adjacent to an additional vertex in the $vxzwu$-face; otherwise this additional neighbor together with t and v would form a claw at x. Also z cannot be adjacent to an additional vertex in the $vxzwu$-face; otherwise this additional neighbor together with t and w would form a claw at z. But then $\{v, w\}$ is a 2-cut, separating u from the rest of the graph and contradicting the fact that G is 3-connected. Thus w and (by a symmetric argument) x, must be adjacent to an additional vertex. In addition, note that by claw-freedom if x (or similarly w) is adjacent to a vertex in the $vwzxy$-face, then x (or similarly w) is not adjacent to a vertex in the $vxzwu$-face (and vice versa). Thus since G is 3-connected and neither $\{v, x\}$ nor $\{v, w\}$ are 2-cuts, one of w and x must be adjacent to an additional vertex in the $vxzwu$-face and the other in the $vwzxy$-face. Thus either x is adjacent to an additional vertex in the $vxzwu$-face and w is adjacent to an additional vertex in the $vwzxy$-face, or w is adjacent to an additional vertex in the $vxzwu$-face and x is adjacent to an additional vertex in the $vwzxy$-face.

Claim 2.4/1.3.4.4.3.1: The vertex w is adjacent to an additional vertex in the $vxzwu$-face and x is adjacent to an additional vertex in the $vwzxy$-face.

Proof of Claim 2.4/1.3.4.4.3.1: By way of contradiction, suppose the vertex x

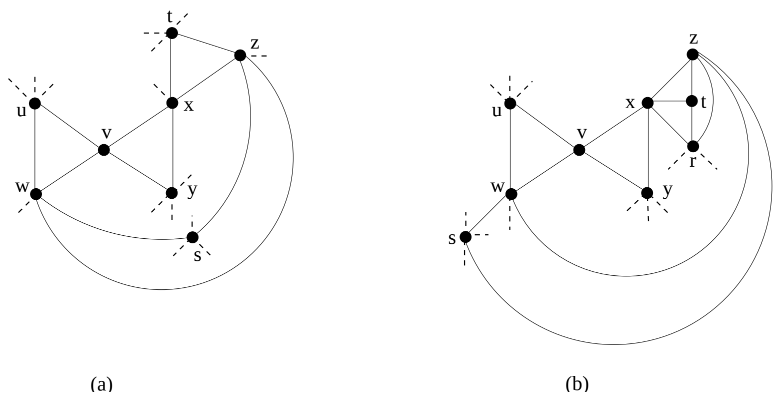


Figure 89: Proving that every vertex of degree four must lie on a K_4.

is adjacent to an additional vertex in the $vxzwu$-face; call it t, and w is adjacent to an additional vertex in the $vwzxy$-face; call it s. To prevent $\{xv, xz, xt\}$ and $\{wv, wz, ws\}$ from forming claws at x and w respectively, we must have $z \sim t$ and $z \sim s$. See Figure 89(a) for an illustration. Since G is 3-connected, $\{v, x\}$ is not a 2-cut, and so there must be a path from y to w and z that does not pass through either v or x. Thus either y is adjacent to s, or z and w share (by claw-freedom) an additional neighbor in the $vwszxy$-face.

Suppose $y \sim s$. Then z has no additional neighbors in the $xysz$-face; otherwise this additional neighbor together with t and w would form a claw at z, contradicting the fact that G is claw-free. Also w has no additional neighbors in the $vysw$-face; otherwise this additional neighbor together with v and z would form a claw at w, contradicting the fact that G is claw-free. Thus by 3-connectivity, y cannot grow and $d(y) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Thus by Lemma 2.2, u must be adjacent to an additional vertex; call it r. Since G is 3-connected, $\{u, t\}$ is not a 2-cut, and so there must be a path from r to x and z that does not pass through either u or t. Note that since $r \nsim z$ by birth, $r \nsim x$ by claw-freedom at x. Thus by claw-freedom, x and z must share an additional neighbor in the $vxzwu$-face; call it q. To prevent $\{xv, xq, xt\}$ from forming a claw at x, we must have $t \sim q$. Note that since q is a vertex on a path between x and r that does not pass through t, vertices t must be interior to the zxq-face. But then $\{u, q\}$ is a 2-cut, separating r from the rest of the graph and contradicting the fact that G is 3-connected. Hence $y \nsim s$.

Suppose z and w share an additional neighbor in the $vwszxy$-face; call it r. To prevent $\{wv, ws, wr\}$ from forming a claw at w, we must have $s \sim r$. Now $d(z) = 5$, and so $d(x) = 4$, since by claw-freedom x may only be adjacent to vertices within the $vxzwu$-face, and if x were adjacent to any additional vertices then the additional neighbor together with v and z would form a claw at x. Since G is 3-connected, $\{u, t\}$ is not a 2-cut, and so there are no additional vertices in the $vxzwu$-face. Thus since $\{v, w\}$ is not a 2-cut, we must have $u \sim t$. Since $d(w) = 5$, s cannot grow by Claim 2.4/1.2. Since $\{y, r\}$ is not a 2-cut, there are no additional vertices in the $vwrzxy$-face. Thus since $\{v, x\}$ is not a 2-cut, we must have $y \sim r$, and the graph can grow no further. But it is not well-covered, since $\{w, x\}$ and $\{w, t, y\}$ are both maximal independent sets of the graph. Hence z and w do not share an additional neighbor in the $vwszxy$-face. Therefore we must have the vertex w adjacent to an additional vertex in the $vxzwu$-face and x adjacent to an additional vertex in the $vwzxy$-face. ■

Claim 2.4/1.3.4.4.3.2: The vertex x is adjacent to an additional vertex in the $vxzwu$-face and w is adjacent to an additional vertex in the $vwzxy$-face.

Proof of Claim 2.4/1.3.4.4.3.2: By way of contradiction, suppose the vertex w is adjacent to an additional vertex in the $vxzwu$-face; call it s, and x is adjacent to an additional vertex in the $vwzxy$-face; call it t. To prevent $\{xv, xz, xt\}$ and $\{wv, wz, ws\}$ from forming claws at x and w respectively, we must have $z \sim t$ and $z \sim s$. Note that $y \nsim t$ otherwise we would have a forbidden subgraph centered at

x by Claim 2.4/1.3.4.1. Thus since $z \nsim y$ by Claim 2.4/1.3.4.4.1 and $\{v, x\}$ is not a 2-cut, there must be another vertex in the $vwzxy$-face. Since $\{y, t\}$ is not a 2-cut, x and z must share another neighbor in this face; call it r. To prevent $\{xv, xt, xr\}$ from forming a claw at x, we must have $t \sim r$. Note that since r is a vertex on a path between x and y that does not pass through t, vertices t must be interior to the zxr-face. See Figure 89(b) for an illustration. Now $d(x) = d(z) = 5$, so x and z cannot grow. Note that $y \nsim r$ otherwise we would have the forbidden subgraph centered at x shown in Figure 5(b). Since G is 3-connected, $\{v, x\}$ is not a 2-cut, and so there must be a path from y to r that does not pass through either v or x. So y must be adjacent to an additional vertex; call it q. But the $\{y, r\}$ is a 2-cut, separating q from the rest of the graph and contradicting the fact that G is 3-connected. Therefore we must have the vertex x adjacent to an additional vertex in the $vxzwu$-face and w adjacent to an additional vertex in the $vwzxy$-face. ■

But Claim 2.4/1.3.4.4.3.1 contradicts Claim 2.4/1.3.4.4.3.2, and so $z \nsim w$. ■

Combining Claims 2.4/1.3.4.4.1, 2.4/1.3.4.4.2, and 2.4/1.3.4.4.3, we have shown that x does not share any of its neighbors outside the bow-tie subgraph with other vertices in the bow-tie subgraph. By symmetric arguments, we may assume this is true for u, w and y as well.

Claim 2.4/1.3.4.4.4: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4: By way of contradiction, suppose y is adjacent to an additional vertex; call it t. Note that $t \nsim z$ by birth. Thus by symmetry, we may assume that any two vertices that have adjacent neighbors that are on a bow-tie are independent. Also note that by Claim 2.4/1.3.4.4.2 and Claim 2.4/1.3.4.4.3, t is adjacent to neither u nor w. Since G is 3-connected, v is not a cut-vertex and so u and w must be adjacent to additional vertices. These additional vertices must be distinct by Claim 2.4/1.3.4.3, since u and w already share the neighbor v, and v cannot grow.

Let s be the additional neighbor of u, and r be the additional neighbor of w. Since $t \nsim z$, by symmetry we may assume that $s \nsim r$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ that contains x, so that $V(C) = \{v, x, z\}$. Recall that by Claim 2.4/1.3.4.4.2 and Claim 2.4/1.3.4.4.3, neither s nor r is adjacent to x. Thus by Lemma 2.2, either s is adjacent to t (or similarly $z \sim r$), r is adjacent to t (or similarly z is adjacent to s), x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.4.1: The vertex s is not adjacent to t.

Proof of Claim 2.4/1.3.4.4.4.1: By way of contradiction, suppose $s \sim t$. See Figure 90(a) for an illustration. Note that $s \nsim z$; otherwise $\{su, sz, st\}$ would be a claw at s, contradicting the fact that G is claw-free. Also note that $r \nsim t$; otherwise $\{ty, ts, tr\}$ would be a claw at t, contradicting the fact that G is claw-free. Since G is 3-connected, $\{x\}$ and $\{w\}$ are not cut-vertices, and so there must be at least one additional vertex in each of the $uvxyts$-face and the $uwvyts$-face. Note that by claw-freedom: if y is adjacent to an additional vertex, then t must be adjacent to that vertex; if u is adjacent to an additional vertex, then s must be adjacent to that vertex; if t is adjacent to an additional vertex, then either y or s is adjacent to that vertex; and if s is adjacent to an additional vertex, then either u or t is adjacent to that vertex. Thus to eliminate the cut-vertices, we have three options to grow to z and r from the set $\{u, y, t, s\}$: y and t share an additional neighbor, u and s share an additional neighbor, or t and s share an additional neighbor. To eliminate both cut-vertices, we need at least one of these options in each of the two faces. At least one of the two faces can have only one of these options, since by claw-freedom each option can occur in only one face. Without loss of generality, suppose the $uwvyts$-face has only one of these options, and suppose y and t share an additional neighbor in the $uwvyts$-face; call it q. By assumption u and s do not have any neighbors in

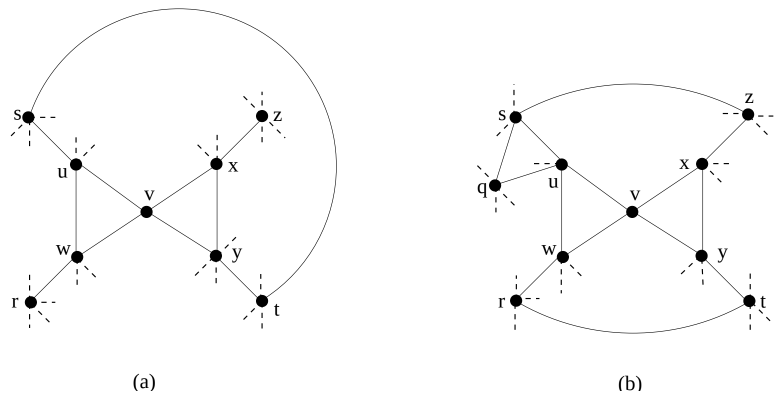


Figure 90: Proving that every vertex of degree four must lie on a K_4.

the $uwvyts$-face. Since G is 3-connected, $\{w, q\}$ is not a 2-cut, and so there must be a path from r to y and t that does not pass through either w or q. Thus y and t must share another neighbor in the $uwvyts$-face; call it p. To prevent $\{yv, yq, yp\}$ from forming a claw at y, we must have $q \sim p$. Note that since p is a vertex on a path between y and r that does not pass through q, the vertex q must be interior to the ytp-face. Now $d(y) = 5$ and so y cannot grow. Thus t cannot have any additional neighbors in the $uwvyts$-face; otherwise the additional neighbor together with y and s would form a claw at t, contradicting the fact that G is claw-free. But then $\{p, w\}$ is a 2-cut, separating r from the rest of the graph and contradicting the fact that G is 3-connected. Thus $s \nsim t$. ■

By a similar argument, we may also assume that $z \nsim r$.

Claim 2.4/1.3.4.4.4.2: The vertex r is not adjacent to t.

Proof of Claim 2.4/1.3.4.4.4.2: By way of contradiction, suppose $r \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either s is adjacent to z, y is adjacent to an additional vertex, x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.4.2.1: The vertex s is not adjacent to z.

Proof of Claim 2.4/1.3.4.4.4.2.1: By way of contradiction, suppose $z \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, z]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $r \nsim z$; otherwise $\{zx, zs, zr\}$ would be a claw at z, contradicting the fact that G is claw-free. Thus by Lemma 2.2, either u is adjacent to an additional vertex or y is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.4.2.1.1: The vertex u is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.1.1: By way of contradiction, suppose u is adjacent to an additional vertex; call it q. To prevent $\{uv, us, uq\}$ from forming a claw at u, we must have $s \sim q$. Note that q could be interior to either the $uvxzs$-face or to the $uwrtyxzs$-face. One of the two possible cases is shown in Figure 90(b). Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that q is adjacent to neither z nor r by birth. Since vertices of the bow-tie do not share neighbors exterior to the bow-tie, q is adjacent to neither x nor y. Thus by Lemma 2.2, either y is adjacent to an additional vertex, x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yp, yt\}$ from forming a claw at y, we must have $t \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing x, so that $V(C) = \{v, x, z\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that p is adjacent to neither q nor r by birth. Also note that since $t \nsim z$ by birth, we may assume by symmetry that $p \nsim z$. Thus by Lemma 2.2, either x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Suppose x is adjacent to an additional vertex; call it n. To prevent $\{xv, xz, xn\}$ from forming a claw at x, we must have $z \sim n$. Note that n may be either in the $vxzxu$-face or the $xytrwusz$-face. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing x, so that $V(C) = \{v, x, z, n\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that n is not adjacent to r, q, or p by birth. Thus by Lemma 2.2, either x is adjacent to an additional vertex, z is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose x is adjacent to an additional vertex; call it m. To prevent $\{xv, xz, xm\}$ and $\{xv, xn, xm\}$ from forming claws at x, we must have $z \sim m$ and $n \sim m$. Either m is interior to the xzn-face, or n is interior to the xzm-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever vertex of m or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p]$ containing x, so that $V(C) = \{v, x, z, n, m\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and one of n and m cannot grow, and v is adjacent to neither n nor m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional fifth vertex and so $d(x) = 4$.

Suppose z is adjacent to an additional vertex; call it m. To prevent $\{zx, zs, zm\}$ from forming a claw at z, we must have $s \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p, m]$ containing x, so that $V(C) = \{v, x, n\}$. Then C is not well-covered, since $\{x\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that m is not adjacent to r, q or p by birth. Also note that $m \nsim n$; otherwise we would have a forbidden subgraph centered at z by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, k]$ containing v, so

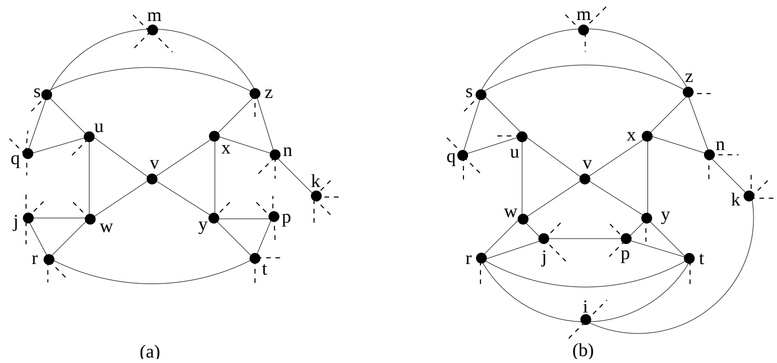


Figure 91: Proving that every vertex of degree four must lie on a K_4.

that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Since k is adjacent to neither p nor r by birth, $k \nsim t$ by claw-freedom at t. Since k is not adjacent to m or q by birth, $k \nsim s$ by claw-freedom at s. Since $k \nsim r$ by birth, $k \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it j. To prevent $\{wv, wr, wj\}$ from forming a claw at w, we must have $r \sim j$. See Figure 91(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n]$ containing w, so that $V(C) = \{v, w, r, j\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that $n \nsim s$; otherwise we would have a forbidden subgraph centered at z, by Claim 2.4/1.3.4.2. Since p is adjacent to neither z nor q, $p \nsim s$ by claw-freedom at s. Since neither p nor n is adjacent to r by their births, neither p nor n is adjacent to w by claw-freedom at w. Note that $j \nsim s$ by birth. Since $j \nsim k$ by birth, $j \nsim n$ by claw-freedom at n. Thus by Lemma 2.2, either j is adjacent to p, w is adjacent to an additional vertex, r is adjacent to an additional vertex, or j is adjacent to an additional vertex.

Suppose $j \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wr, wi\}$ and $\{wv, wj, wi\}$ from forming claws at w, we must have $r \sim i$ and $j \sim i$. Thus by planarity, i must be interior to the wrj-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, i cannot grow and $d(i) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n]$ containing w, so that $V(C) = \{v, w, r, i\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and i cannot grow, and $v \nsim i$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $d(w) = 4$.

Suppose r is adjacent to an additional vertex; call it i. To prevent $\{rw, rt, ri\}$ from forming a claw at r, we must have $t \sim i$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, k, i]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that i is not adjacent to either s or p by birth, and C cannot grow. Thus by Lemma 2.2, we must have $i \sim k$. See Figure 91(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, k, j]$ containing y, so that $V(C) = \{v, x, y, t\}$. Then C is not well-covered, since $\{y\}$ and $\{v, t\}$ are both maximal independent sets of C. Recall that $j \nsim k$ by birth. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or t is adjacent to an additional vertex. Suppose y is adjacent to an additional vertex; call it h. To prevent $\{yv, yt, yh\}$ and $\{yv, yp, yh\}$ from forming claws at y, we must have $t \sim h$ and $p \sim h$. Thus by planarity, h must be interior to the ytp-face. Now $d(y) = d(t) = 5$ and so by Claim 2.4/1.2, h cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[s, k, j]$ containing y, so that $V(C) = \{v, x, y, t, h\}$. Then C is not well-covered, since $\{y\}$ and $\{v, h\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence y is not adjacent to an additional vertex and so $d(y) = 4$. Suppose t is adjacent to an additional vertex; call it h. To prevent $\{ty, tr, th\}$ and $\{ty, ti, th\}$ from forming claws at t, we must have $r \sim h$ and $i \sim h$. Thus by planarity, h must be interior to the tri-face. Now $d(t) = d(r) = 5$ and so by Claim 2.4/1.2, h cannot grow and $d(h) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, k, h]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence, t is not adjacent to an additional vertex and so r is not adjacent to an additional vertex. Therefore $j \nsim p$.

Suppose w is adjacent to an additional vertex; call it i. To prevent $\{wv, wr, wi\}$ and $\{wv, wj, wi\}$ from forming claws at w, we must have $r \sim i$ and $j \sim i$. Either i is interior to the wrj-face, or j is interior to the wri-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, whichever vertex of i or j is interior cannot grow and must have degree three. See Figure 92(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n]$ containing w, so that $V(C) = \{v, w, r, j, i\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of i or j cannot grow, and v is adjacent to neither i nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex and so $d(w) = 4$.

Suppose r is adjacent to an additional vertex; call it i. To prevent $\{rw, rt, ri\}$ from forming a claw at r, we must have $i \sim t$. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n, i]$ containing w, so that $V(C) = \{v, w, j\}$. Then C is not well-covered, since $\{w\}$ and $\{v, j\}$ are both maximal independent sets of C. Note that i is not adjacent to s, p or n by birth. Also note that $i \nsim j$ by Claim 2.4/1.3.4.3, since r and j already share the neighbor w and w cannot grow. Thus by Lemma 2.2, j must be adjacent to an additional vertex; call it h. Call this semi-

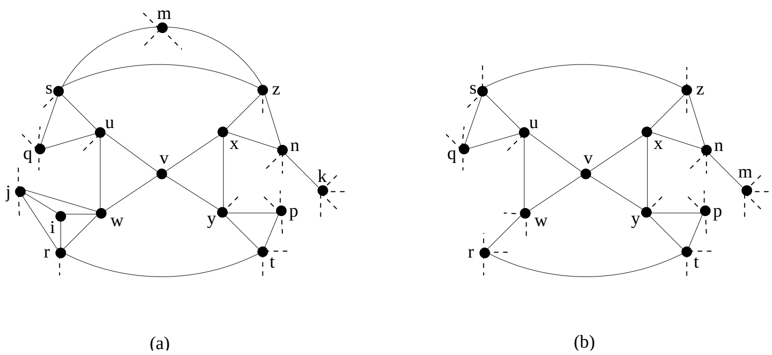


Figure 92: Proving that every vertex of degree four must lie on a K_4.

known subgraph S. Let C be the component of $S - N[t, s, k, h]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $h \nsim s$ by birth. Since h is adjacent to neither i nor p by birth, $h \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, we must have $h \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, q, k]$ containing w, so that $V(C) = \{v, w, j\}$. Then C is not well-covered, since $\{w\}$ and $\{v, j\}$ are both maximal independent sets of C. Note that $k \nsim z$ by Claim 2.4/1.3.4.3, since z and n already share the neighbor x and x cannot grow. Recall that j is adjacent to neither k nor t by birth. Since $j \nsim s$ by birth, $j \nsim z$ by claw-freedom at z. Also note that $q \nsim j$ since neighbors of different vertices on the same side of a bow-tie (i.e. u and w) do not have edges between them. Thus by Lemma 2.2, j must be adjacent to an additional vertex; call it g. To prevent $\{jw, jh, jg\}$ from forming a claw at j, we must have $g \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, k, g]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that g is adjacent to neither k nor t by birth. Since g is adjacent to neither z nor q by birth, $g \nsim s$ by claw-freedom at s. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex.

Suppose j is adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[s, p, n, i]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that i is not adjacent to s, p, or n by birth. Recall that r is not adjacent to an additional vertex, which includes i, since we concluded that r could not be adjacent to an additional vertex *before* i was born. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence j is not adjacent to an additional vertex and so z is not adjacent to an additional vertex.

Suppose n is adjacent to an additional vertex; call it m. See Figure 92(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, p, m]$ containing x, so that $V(C) = \{v, x, z\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that m is not adjacent to r, q or p by birth. Recall that z is not adjacent to an additional vertex, which includes m, since we concluded that z could not be adjacent to an additional vertex *before* m was born. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so x is not adjacent to an additional vertex. Therefore $d(x) = 3$.

Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zx, zs, zn\}$ from forming a claw at z, we must have $s \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that s and t are not adjacent to w since vertices on a bow-tie do not share neighbors. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it m. To prevent $\{wv, wr, wm\}$ from forming a claw at w, we must have $r \sim m$. See Figure 93(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing u, so that $V(C) = \{u, v, q\}$. Then C is not

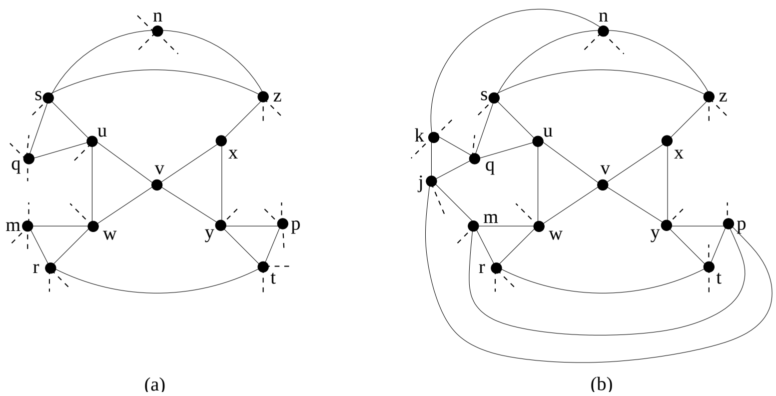


Figure 93: Proving that every vertex of degree four must lie on a K_4.

well-covered, since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Since $r \nsim s$, $r \nsim z$ by claw-freedom at z. Recall that $p \nsim r$ by birth, and $q \nsim z$ by birth. Thus by Lemma 2.2, either u is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it k. To prevent $\{uv, us, uk\}$ and $\{uv, uq, uk\}$ from forming claws at u, we must have $s \sim k$ and $q \sim k$. Either k is interior to the usq-face, or q is interior to the usk-face. Now $d(u) = d(s) = 5$ and so by Claim 2.4/1.2, whichever of k or q is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing u, so that $V(C) = \{u, v, q, k\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and one of k and q cannot grow, and v is adjacent to neither q nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence u is not adjacent to an additional vertex, and $d(u) = 4$.

Suppose q is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[r, p, n, k]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Note that k is adjacent to neither r nor p by birth. Thus by

Lemma 2.2, we must have $k \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, z, r]$ containing q, so that $V(C) = \{u, q, k\}$. Then C is not well-covered since $\{q\}$ and $\{u, k\}$ are both maximal independent sets of C. Note that k is not adjacent to r or z by birth. Since $k \nsim p$ by birth, $k \nsim y$ by claw-freedom at y. Thus by Lemma 2.2, either q is adjacent to an additional vertex, or k is adjacent to an additional vertex.

Suppose q is adjacent to an additional vertex; call it j. To prevent $\{qu, qk, qj\}$ from forming a claw at q, we must have $k \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[r, p, n, j]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Note that $j \nsim r$ by birth. Also note that $n \nsim j$ by Claim 2.4/1.3.4.3, since k and j already share the neighbor q, and $n \nsim q$ by birth. Thus by Lemma 2.2, we must have $j \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, m, j]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Since j is adjacent to neither r nor y by birth, $j \nsim t$ by claw-freedom at t. Recall that $m \nsim t$ by birth. Thus by Lemma 2.2, we must have $j \sim m$. To prevent $\{jq, jp, jm\}$ from forming a claw at j, we must have $m \sim p$. See Figure 93(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[w, t, k]$ containing z, so that $V(C) = \{x, z, s\}$. Then C is not well-covered since $\{z\}$ and $\{x, s\}$ are both maximal independent sets of C. Note that k is adjacent to neither t nor w by planarity. Thus by Lemma 2.2 and claw-freedom, z and s must share an additional neighbor, call it i. To prevent $\{zx, zn, zi\}$ from forming a claw at z, we must have $i \sim n$. Thus by planarity, i must be interior to the zsn-face. Now $d(s) = 5$, and so by Claim 2.4/1.2, i cannot grow and $d(i) = 3$. Also note that z cannot be adjacent to an additional vertex; otherwise the additional neighbor together with s and x would form a claw at z, contradicting the fact that G is claw-free. Call this semi-known subgraph S.

Let C be the component of $S - N[w, t, k]$ containing z, so that $V(C) = \{x, z, s, i\}$. Then C is not well-covered since $\{z\}$ and $\{x, s\}$ are both maximal independent sets of C. Note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex.

Suppose k is adjacent to an additional vertex; call it j. To prevent $\{kq, kn, kj\}$ from forming a claw at k, we must have $k \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p, j]$ containing u, so that $V(C) = \{u, v, q\}$. Then C is not well-covered since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that j is adjacent to neither z nor r by birth. Recall that q is not adjacent to an additional vertex, which includes j, by the preceding case. Thus by Lemma 2.2, we must have $j \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, m, j]$ containing u, so that $V(C) = \{u, v, q\}$. Then C is not well-covered since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Since j is adjacent to neither y nor r by birth, $j \nsim t$ by claw-freedom at t. Recall that $m \nsim t$ by birth. Since $q \nsim r$ by birth, $q \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either j is adjacent to m, or m is adjacent to q. Suppose $j \sim m$. To prevent $\{jp, jn, jm\}$ from forming a claw at j, we must have $m \sim p$ since $n \nsim m$ by planarity. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, j]$ containing u, so that $V(C) = \{u, v, w, q\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and q cannot grow, and $u \nsim q$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $m \nsim j$. Suppose $m \sim q$. To prevent $\{qu, qk, qm\}$ from forming a claw at q, we must have $m \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[p, n, m]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Since p is adjacent to neither r nor k, $p \nsim m$ by claw-freedom at m. Recall that n is not adjacent to p by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim q$, and so k is not adjacent to an additional vertex. Therefore q is not

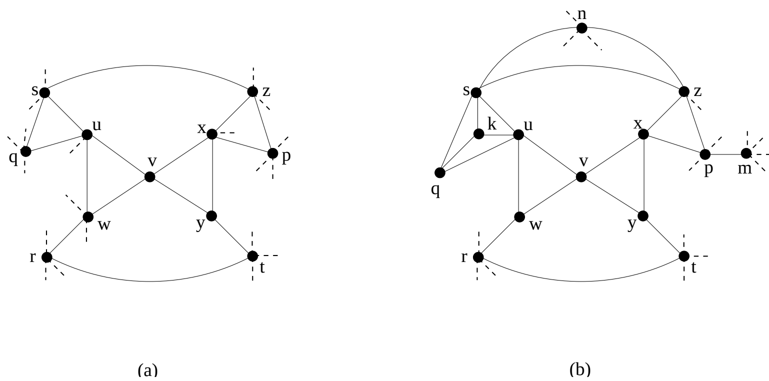


Figure 94: Proving that every vertex of degree four must lie on a K_4.

adjacent to an additional vertex and so z is not adjacent to an additional vertex. Thus y is not adjacent to an additional vertex and $d(y) = 3$.

Suppose x is adjacent to an additional vertex; call it p. To prevent $\{xv, xz, xp\}$ from forming a claw at x, we must have $z \sim p$. The vertex p is either interior or exterior to the $uvxzs$-face. One of the two possible cases is shown in Figure 94(a). Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing x, so that $V(C) = \{v, x, y, z, p\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that p is adjacent to neither q nor r by birth. Thus by Lemma 2.2, either x is adjacent to an additional vertex, z is adjacent to an additional vertex, or p is adjacent to an additional vertex.

Suppose x is adjacent to an additional vertex; call it n. To prevent $\{xv, xz, xn\}$ and $\{xv, xp, xn\}$ from forming claws at x, we must have $z \sim n$ and $p \sim n$. Either n is interior to the usq-face, or q is interior to the usn-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever of n or q is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[r, q]$ containing x, so that $V(C) = \{v, x, y, z, p, n\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v, and one of p and n cannot grow, and v is adjacent to neither p nor n. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex, and $d(x) = 5$.

Suppose z is adjacent to an additional vertex; call it n. To prevent $\{zx, zs, zn\}$ from forming a claw at z, we must have $n \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, n]$ containing x, so that $V(C) = \{v, x, y, p\}$. Then C is not well-covered since $\{x\}$ and $\{v, p\}$ are both maximal independent sets of C. Note that n is adjacent to neither q nor r by birth. Also note that $n \nsim p$ by Claim 2.4/1.3.4.3, since z and p already share the neighbor x, and x cannot grow. Thus by Lemma 2.2, p must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, m]$ containing u, so that $V(C) = \{u, v, w, q\}$. Then C is not well-covered since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that m is not adjacent to q by birth. Since $m \nsim r$ by birth, $m \nsim t$ by claw-freedom at t. Also note that $m \nsim z$ by Claim 2.4/1.3.4.3, since z and p already share the neighbor x, and x cannot grow. Thus by Lemma 2.2, either w is adjacent to an additional vertex, u is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wr, wk\}$ from forming a claw at w, we must have $r \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, m, k]$ containing u, so that $V(C) = \{u, v, q\}$. Then C is not well-covered since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that k is not adjacent to z, t or m by birth. Also note that $k \nsim u$ otherwise we would have a forbidden subgraph centered at w by Claim 2.4/1.3.4.1. Finally, note that $k \nsim q$, since k and q are neighbors of vertices on the same side of the bow-tie, and thus cannot be adjacent. Thus by Lemma 2.2 either u is adjacent to an additional vertex or q is adjacent to an additional vertex. Suppose u is adjacent to an additional vertex; call it j. To prevent $\{uv, us, uj\}$ and $\{uv, uq, uj\}$ from forming claws at u, we must have $s \sim j$ and $q \sim j$. Either j is interior to the usq-face, or q

is interior to the usj-face. Now $d(u) = d(s) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, m, k]$ containing u, so that $V(C) = \{u, v, q, j\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and one of j or q cannot grow, and v is adjacent to neither q nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus u is not adjacent to an additional vertex and so $d(u) = 4$. Suppose q is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, j]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of G. Note that $j \nsim z$ by birth. Since j is adjacent to neither k nor t by birth, $j \nsim r$ by claw-freedom at r. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so w is not adjacent to an additional vertex and so $d(w) = 3$.

Suppose u is adjacent to an additional vertex; call it k. To prevent $\{uv, us, uk\}$ and $\{uv, uq, uk\}$ from forming claws at u, we must have $s \sim k$ and $q \sim k$. Either k is interior to the usq-face, or q is interior to the usk-face. Now $d(u) = d(s) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. One of the two possible cases is shown in Figure 94(b). Call this semi-known subgraph S. Let C be the component of $S - N[z, t, m]$ containing u, so that $V(C) = \{u, v, w, q, k\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and one of k or q cannot grow, and v is adjacent to neither q nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus u is not adjacent to an additional vertex and so $d(u) = 4$.

Suppose q is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, m, k]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that k is adjacent to neither z nor m by birth. Since $k \nsim t$ by birth, $k \nsim r$ by claw-freedom at r. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so z is not adjacent to an additional vertex. Therefore $d(z) = 3$.

Suppose p is adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[r, q, n]$ containing x, so that $V(C) = \{v, x, y, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that n is not adjacent to either r or q by birth, and C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence p is not adjacent to an additional vertex and so x is not adjacent to an additional vertex. Therefore $d(x) = 3$.

Suppose z is adjacent to an additional vertex; call it p. To prevent $\{zx, zs, zp\}$ from forming a claw at z, we must have $s \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Recall that $s \nsim t$ by Claim 2.4/1.3.4.4.4.1. Thus by Lemma 2.2, w must be adjacent to an additional vertex; call it n. To prevent $\{wv, wr, wn\}$ from forming a claw at w, we must have $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[s, n]$ containing y, so that $V(C) = \{v, x, y, t\}$. Then C is not well-covered since $\{y\}$ and $\{x, t\}$ are both maximal independent sets of C. Note that n is not adjacent to t or s by birth. Thus by Lemma 2.2, t must be adjacent to an additional vertex; call it m. To prevent $\{ty, tr, tm\}$ from forming a claw at t, we must have $r \sim m$. See Figure 95(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, n]$ containing u, so that $V(C) = \{u, v, q\}$. Then C is not well-covered since $\{u\}$ and $\{v, q\}$ are both maximal independent sets of C. Since $n \nsim s$ by birth, $n \nsim z$ by claw-freedom at z. Note that $n \nsim t$ by birth. Also note

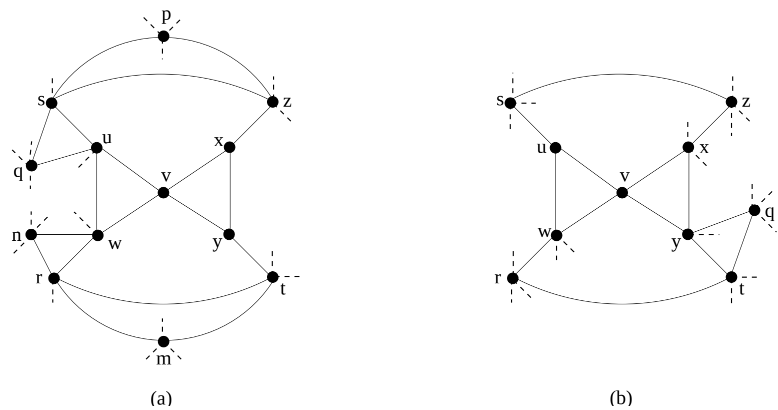


Figure 95: Proving that every vertex of degree four must lie on a K_4.

that $n \nsim q$, since n and q are neighbors of vertices on the same side of the bow-tie, and thus cannot be adjacent. Since $q \nsim r$ by birth, $q \nsim t$ by claw-freedom at t. Thus by Lemma 2.2, either u is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it k. To prevent $\{uv, us, uk\}$ and $\{uv, uq, uk\}$ from forming claws at u, we must have $s \sim k$ and $q \sim k$. Either k is interior to the usq-face, or q is interior to the usk-face. Now $d(u) = d(s) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, n]$ containing u, so that $V(C) = \{u, v, q, k\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and one of q and k cannot grow, and v is adjacent to neither q nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence u is not adjacent to an additional vertex and so $d(u) = 4$.

Suppose q is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, k]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $k \nsim z$ by birth. Since k is adjacent to neither t nor n

by birth, $k \nsim r$ by claw-freedom at r. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so z is not adjacent to an additional vertex. Therefore u is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.4.4.2.1.1. ■

By Claim 2.4/1.3.4.4.4.2.1.1, we may assume that $d(u) = 3$.

Claim 2.4/1.3.4.4.4.2.1.2: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.1.2: By way of contradiction, suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, yt, yq\}$ from forming a claw at y, we must have $q \sim t$. See Figure 95(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing y, so that $V(C) = \{v, x, y, q\}$. Then C is not well-covered, since $\{y\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that $q \nsim r$ by birth. Since $q \nsim z$ by birth, $q \nsim s$ by claw-freedom at s. Thus by Lemma 2.2, either y is adjacent to an additional vertex, x is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yt, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $t \sim p$ and $q \sim p$. Either p is interior to the ytq-face, or q is interior to the ytp-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing y, so that $V(C) = \{v, x, y, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and one of q and p cannot grow, and v is adjacent to neither q nor p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus y is not adjacent to an additional vertex and so $d(y) = 4$.

Suppose x is adjacent to an additional vertex; call it p. To prevent $\{xv, xz, xp\}$ from forming a claw at x, we must have $z \sim p$. Call this semi-known subgraph S.

Let C be the component of $S - N[t, p]$ containing u, so that $V(C) = \{u, v, w, s\}$. Then C is not well-covered since $\{u\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that $t \nsim p$ since x and y are adjacent on a bow-tie, and so their neighbors cannot be adjacent. Also note that $p \nsim s$ by birth. Since $p \nsim r$ by birth, $p \nsim w$ by claw-freedom at w. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or s is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wr, wn\}$ from forming a claw at w, we must have $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[t, p, n]$ containing u, so that $V(C) = \{u, v, s\}$. Then C is not well-covered since $\{u\}$ and $\{v, s\}$ are both maximal independent sets of C. Note that n is not adjacent to either t or p by birth. Thus by Lemma 2.2, s must be adjacent to an additional vertex; call it m. To prevent $\{su, sz, sm\}$ from forming a claw at s, we must have $z \sim m$. See Figure 96(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, m]$ containing w, so that $V(C) = \{u, w, r, n\}$. Then C is not well-covered since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Note that m is adjacent to neither p nor n by birth. Since $m \nsim n$ by birth, $m \nsim w$ by claw-freedom at w. Since m is adjacent to neither t nor n by birth, $m \nsim r$ by claw-freedom at r. Thus by Lemma 2.2, either w is adjacent to an additional vertex, r is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wr, wk\}$ and $\{wv, wn, wk\}$ from forming claws at w, we must have $r \sim k$ and $n \sim k$. Either k is interior to the wrn-face, or n is interior to the wrk-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever of k or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, m]$ containing w, so that $V(C) = \{u, w, r, n, k\}$. Then every vertex of $C - w$ is adjacent

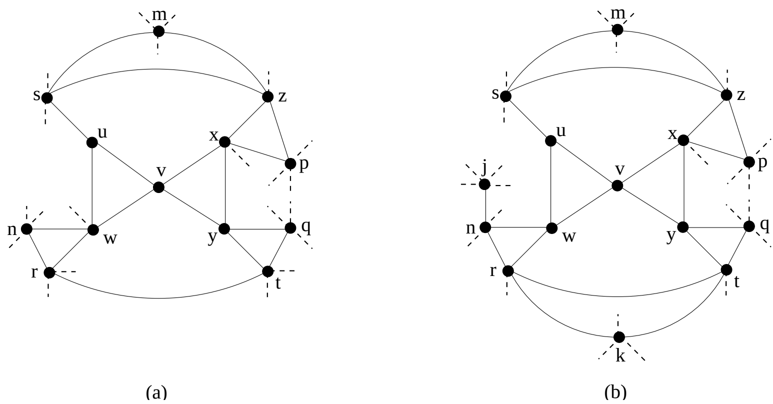


Figure 96: Proving that every vertex of degree four must lie on a K_4.

to w, vertices w, u and one of n and k cannot grow, and u is adjacent to neither n nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional fifth vertex and so $d(w) = 4$.

Suppose r is adjacent to an additional vertex; call it k. To prevent $\{rw, rt, rk\}$ from forming a claw at r, we must have $t \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, m, k]$ containing w, so that $V(C) = \{u, w, n\}$. Then C is not well-covered, since $\{w\}$ and $\{u, n\}$ are both maximal independent sets of C. Note that k is not adjacent to y, p or m by birth. Also note that $n \nsim k$ otherwise we would have a forbidden subgraph centered at r by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it j. See Figure 96(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, s, p, j]$ containing r, so that $V(C) = \{w, r, k\}$. Then C is not well-covered, since $\{r\}$ and $\{w, k\}$ are both maximal independent sets of C. Note that j is not adjacent to y, p, or k by birth. Since neither j nor k is adjacent to m by their births, neither j nor k is adjacent to s by claw-freedom at s. Note that $j \nsim r$ by Claim 2.4/1.3.4.3, since n and r already share the neighbor w, and w cannot grow. Thus by Lemma 2.2, either r is adjacent to an additional vertex, or k is adjacent to an additional vertex. Suppose r is adjacent to an additional vertex;

call it i. To prevent $\{rw, rt, ri\}$ and $\{rw, rk, ri\}$ from forming claws at r, we must have $t \sim i$ and $k \sim i$. Either i is interior to the rtk-face, or k is interior to the rti-face. Now $d(r) = d(t) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, s, p, j]$ containing r, so that $V(C) = \{w, r, k, i\}$. Then every vertex of $C - r$ is adjacent to r, vertices r, w and one of k and i cannot grow, and w is adjacent to neither k nor i. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex. Suppose k is adjacent to an additional vertex; call it i. Call this semi-known subgraph S. Let C be the component of $S - N[y, z, j, i]$ containing w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Note that i is adjacent to neither j nor y by birth. Since i is adjacent to neither s nor p by birth, $i \nsim z$ by claw-freedom at z. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus k is not adjacent to an additional vertex and so r is not adjacent to an additional vertex.

Suppose n is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[y, p, m, k]$ containing w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Note that k is not adjacent to y, p or m by birth. Also note that C cannot grow since r is not adjacent to an additional vertex, which includes k. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Therefore $d(w) = 3$.

Suppose s is adjacent to an additional vertex; call it n. To prevent $\{su, sz, sn\}$ from forming a claw at s, we must have $z \sim n$. See Figure 97(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, z]$ containing

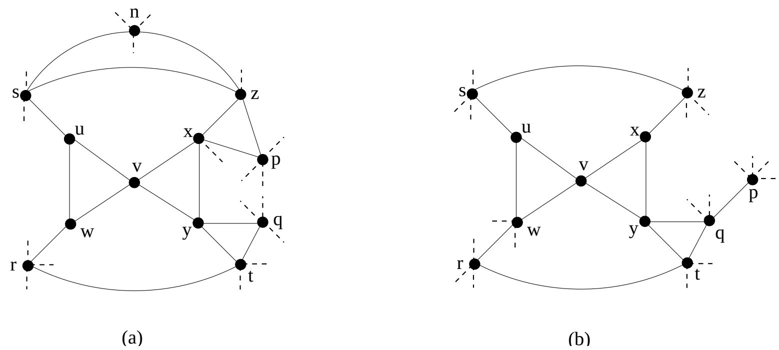


Figure 97: Proving that every vertex of degree four must lie on a K_4.

w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{rw, rt, rm\}$ from forming a claw at r, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q]$ containing r, so that $V(C) = \{w, r, m\}$. Then C is not well-covered, since $\{r\}$ and $\{w, m\}$ are both maximal independent sets of C. Since $m \nsim z$ by birth, m is adjacent to neither x nor s by claw-freedom at x and s respectively. Note that $m \nsim q$ otherwise we would have a forbidden subgraph centered at t by Claim 2.4/1.3.4.1. Thus by Lemma 2.2, either r is adjacent to an additional vertex, or m is adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it k. To prevent $\{rw, rt, rk\}$ and $\{rw, rm, rk\}$ from forming a claw at r, we must have $t \sim k$ and $m \sim k$. Either k is interior to the rtm-face, or m is interior to the rtk-face. Now $d(t) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Note also that $d(r) = 4$ otherwise an additional neighbor together with w and t would form a claw at r (recall that $q \nsim r$ by birth). Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q]$ containing r, so that $V(C) = \{w, r, m, k\}$. Then every vertex of $C - r$ is adjacent to r, vertices r, w and one of m or k cannot grow, and w is adjacent to neither m nor k. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex.

Suppose m is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[y, z, k]$ containing w, so that $V(C) = \{u, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Since k is adjacent to neither x nor s by birth, $k \nsim z$ by claw-freedom at z. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex and so s is not adjacent to an additional vertex. Therefore x is not adjacent to an additional vertex and so $d(x) = 4$.

Suppose q is adjacent to an additional vertex; call it p. See Figure 97(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, r, p]$ containing v, so that $V(C) = \{u, v, y\}$. Then C is not well-covered, since $\{v\}$ and $\{u, y\}$ are both maximal independent sets of C. Note that $p \nsim r$ by birth. Since $p \nsim s$ by birth, $p \nsim z$ by claw-freedom at z. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence q is not adjacent to an additional vertex and so y is not adjacent to an additional vertex. ■

Since neither u nor y are adjacent to additional vertices, $s \nsim z$. Thus we have proved Claim 2.4/1.3.4.4.4.2.1. ■

Claim 2.4/1.3.4.4.4.2.2: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.2: By way of contradiction, suppose y is adjacent to an additional vertex; call it q. To prevent $\{yv, yt, yq\}$ from forming a claw at y, we must have $t \sim q$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, r]$ containing y, so that $V(C) = \{v, y, q\}$. Then C is not well-covered, since $\{y\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that q is adjacent to neither r nor s by birth. Recall that none of the neighbors of x are adjacent to

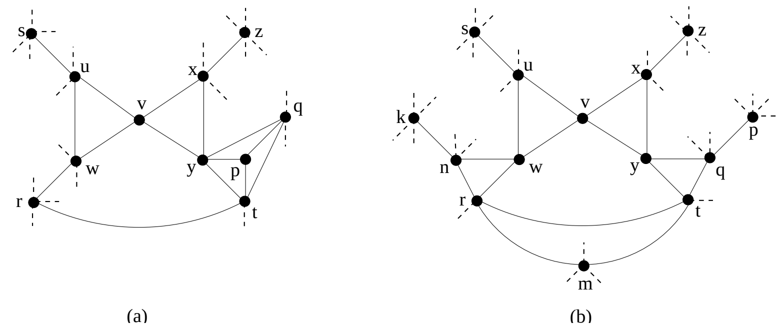


Figure 98: Proving that every vertex of degree four must lie on a K_4.

neighbors of y, and so $q \nsim z$. Thus by Lemma 2.2, either y is adjacent to an additional vertex, or q is adjacent to an additional vertex.

Claim 2.4/1.3.4.4.4.2.2.1: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.2.1: By way of contradiction, suppose y is adjacent to an additional vertex; call it p. To prevent $\{yv, yt, yp\}$ and $\{yv, yq, yp\}$ from forming claws at y, we must have $t \sim p$ and $q \sim p$. Either p is interior to the ytq-face, or q is interior to the ytp-face. Now $d(y) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. One of the two possible cases is shown in Figure 98(a). Call this semi-known subgraph S. Let C be the component of $S - N[z, s, r]$ containing y, so that $V(C) = \{v, y, q, p\}$. Then every vertex of $C - y$ is adjacent to y, vertices y, v and one of q and p cannot grow, and v is adjacent to neither q nor p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus y is not adjacent to an additional vertex. ■

By Claim 2.4/1.3.4.4.4.2.2.1, we may assume that $d(y) = 4$.

Claim 2.4/1.3.4.4.4.2.2.2: The vertex q is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.2.2: By way of contradiction, suppose q is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Recall that q is adjacent to neither r nor s by birth. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it n. To prevent $\{wv, wr, wn\}$ from forming a claw at w, we must have $r \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q]$ containing w, so that $V(C) = \{v, w, r, n\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that n is not adjacent to z, s or q by birth. Thus by Lemma 2.2, either w is adjacent to an additional vertex, r is adjacent to an additional vertex, or n is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wr, wm\}$ and $\{wv, wn, wm\}$ from forming claws at w, we must have $r \sim m$ and $n \sim m$. Either m is interior to the wrn-face, or n is interior to the wrm-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever of m or n is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q]$ containing w, so that $V(C) = \{v, w, r, n, m\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of n and m cannot grow, and v is adjacent to neither n nor m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to a fifth additional vertex and so $d(w) = 4$.

Suppose r is adjacent to an additional vertex; call it m. To prevent $\{rw, rt, rm\}$ from forming a claw at r, we must have $t \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, m]$ containing w, so that $V(C) = \{v, w, n\}$. Then C is not well-covered, since $\{w\}$ and $\{v, n\}$ are both maximal independent sets of C. Note that m is not adjacent to z, s and q by birth. Also note that $m \nsim n$ otherwise we would have a forbidden subgraph centered at r by Claim 2.4/1.3.4.1.

Thus by Lemma 2.2, n must be adjacent to an additional vertex; call it k. See Figure 98(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, p, m, k]$ containing v, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{v\}$ and $\{w, y\}$ are both maximal independent sets of C. Note that k is not adjacent to z, s or m by birth. Recall that p is not adjacent to s or z by birth. Thus by Lemma 2.2, either m is adjacent to p, or k is adjacent to p.

Suppose that $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, p, k]$ containing r, so that $V(C) = \{w, t, r\}$. Then C is not well-covered, since $\{r\}$ and $\{w, t\}$ are both maximal independent sets of C. Since neither p nor k is adjacent to z by birth, neither p nor k is adjacent to x by claw-freedom at x. Since k is adjacent to neither m nor q by birth, $k \nsim p$ by claw-freedom at p. Note that if r is adjacent to an additional vertex, by claw-freedom t must be adjacent to that vertex, and vice versa. Thus by Lemma 2.2, r and t must share an additional vertex; call it j. By planarity, j must be in the rtm-face. To prevent $\{rw, rj, rm\}$ from forming a claw at r, we must have $j \sim m$. Note that $d(y) = d(t) = 5$, and so by Claim 2.4/1.2, j cannot grow and $d(j) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, p, k]$ containing r, so that $V(C) = \{w, t, r, j\}$. Then C is not well-covered, since $\{r\}$ and $\{w, t\}$ are both maximal independent sets of C. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence $m \nsim p$.

Suppose $k \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q, k]$ containing r, so that $V(C) = \{w, r, m\}$. Then C is not well-covered, since $\{r\}$ and $\{w, m\}$ are both maximal independent sets of C. Recall that k is adjacent to neither q nor m by birth. Thus by Lemma 2.2, either r is adjacent to an additional vertex, or m is adjacent to an additional vertex. Suppose r is adjacent to an additional vertex; call it j. To prevent $\{rw, rt, rj\}$ and $\{rw, rj, rm\}$ from forming claws at r, we must have $j \sim t$ and $j \sim m$. Either j is interior to the rtm-face, or m is interior to the rtj-face. Now $d(y) = d(t) = 5$, and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q, k]$ containing r, so that $V(C) = \{w, r, m, j\}$. Then every vertex of $C - r$ is adjacent to r, vertices r, w and one of m and j cannot grow, and w is adjacent to neither m nor j. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $d(r) = 4$. Suppose m is adjacent to an additional vertex; call it j. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, k, j]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that j is not adjacent to s, q or k by birth. Also note that $j \nsim z$ by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence m is not adjacent to an additional vertex and so $k \nsim p$. Therefore r is not adjacent to an additional vertex and so $d(r) = 3$.

Suppose n is adjacent to an additional vertex; call it m. See Figure 99(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, m]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that m is not adjacent to z, s or q by birth. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so w is not adjacent to an additional vertex. Thus $d(w) = 3$.

Suppose r is adjacent to an additional vertex; call it n. To prevent $\{rw, rt, rn\}$ from forming a claw at r, we must have $t \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q]$ containing r, so that $V(C) = \{w, r, n\}$. Then C is not well-covered, since $\{r\}$ and $\{w, n\}$ are both maximal independent sets of C. Note that n is not adjacent to s or q by birth. Since $n \nsim z$ by birth, $n \nsim x$ by claw-freedom at x. Thus by Lemma 2.2, either r is adjacent to an additional vertex,

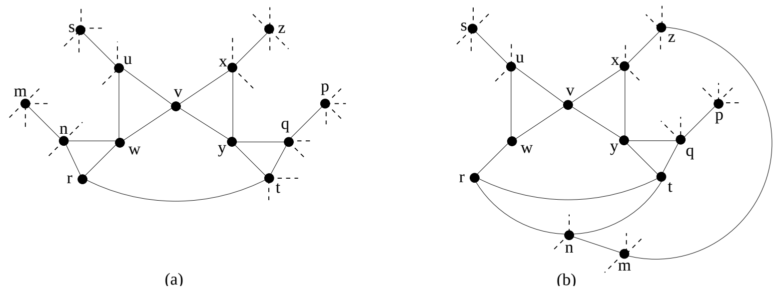


Figure 99: Proving that every vertex of degree four must lie on a K_4.

or n is adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it m. To prevent $\{rw, rt, rm\}$ and $\{rw, rn, rm\}$ from forming claws at r, we must have $t \sim m$ and $n \sim m$. Either m is interior to the rtn-face, or n is interior to the rtm-face. Now $d(t) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Also note that $d(r) = 4$; otherwise an additional neighbor together with w and t would form a claw at r, contradicting the fact that G is claw-free (recall that $r \nsim q$ by birth). Call this semi-known subgraph S. Let C be the component of $S - N[x, s, q]$ containing r, so that $V(C) = \{w, r, n, m\}$. Then every vertex of $C - r$ is adjacent to r, vertices r, w and one of n and m cannot grow, and w is adjacent to neither n nor m. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex and so $d(r) = 3$. Note that this also means that $d(t) = 4$, since if t were adjacent to an additional vertex, this additional neighbor together with r and y would form a claw at t, contradicting the fact that G is claw-free.

Suppose n is adjacent to an additional vertex, and call it m. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, m]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that m is not adjacent to s or q by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $m \sim z$. See Figure 99(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u, z, p]$ containing t, so that $V(C) = \{y, t, r, n\}$. Then every vertex of $C - t$ is adjacent to t, vertices t, y and r cannot grow, and $y \nsim r$. Since $p \nsim s$ by birth, $p \nsim u$ by claw-freedom at u. Recall that $p \nsim z$ by birth. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence n is not adjacent to an additional vertex and so r is not adjacent to an additional vertex. Therefore q is not adjacent to an additional vertex. ■

Since y is not adjacent to a fifth vertex, and q is not adjacent to an additional vertex, y is not adjacent to a fourth vertex. ■

By Claim 2.4/1.3.4.4.4.2.2, we may assume that $d(y) = 3$, and by symmetry, we may assume that $d(w) = 3$ as well.

Claim 2.4/1.3.4.4.4.2.3: The vertex x is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.3: By way of contradiction, suppose x is adjacent to an additional vertex; call it q. To prevent $\{xv, xz, xq\}$ from forming a claw at x, we must have $z \sim q$. See Figure 100(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing x, so that $V(C) = \{v, x, y, z, q\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that q is adjacent to neither r nor s by birth. Thus by Lemma 2.2, either x is adjacent to an additional vertex, or z or q is adjacent to an additional vertex (these two cases are symmetric).

Suppose x is adjacent to an additional vertex; call it p. To prevent $\{xv, xz, xp\}$ and $\{xv, xq, xp\}$ from forming claws at x, we must have $z \sim p$ and $q \sim p$. Either p is interior to the xzq-face, q is interior to the xzp-face, or z is interior to the xqp-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot

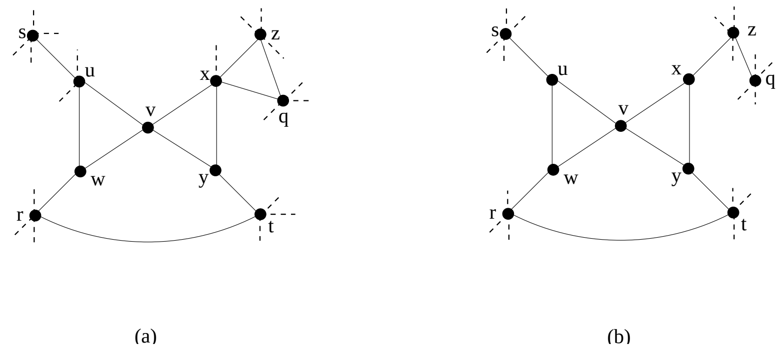


Figure 100: Proving that every vertex of degree four must lie on a K_4.

grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[r, s]$ containing x, so that $V(C) = \{v, x, y, z, q, p\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and one of z, q and p cannot grow, and v is adjacent to none of z, q or p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex and so $d(x) = 4$.

Suppose, without loss of generality, that z is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[s, r, p]$ containing x, so that $V(C) = \{v, x, y, q\}$. Then C is not well-covered, since $\{x\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that p is not adjacent to s or r by birth. Also note that $p \nsim q$ by Claim 2.4/1.3.4.3, since z and q already share the neighbor x and x cannot grow. Thus by Lemma 2.2, q must be adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, p, n]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that n is not adjacent to p or s by birth. Since neither p nor n is adjacent to r by their births, neither p nor n is adjacent to t by claw-freedom at t. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither z nor q is adjacent to an additional vertex.

Therefore x is not adjacent to an additional vertex. ■

By Claim 2.4/1.3.4.4.4.2.3, we may assume that $d(x) = 3$, and by symmetry, we may assume that $d(u) = 3$ as well.

Claim 2.4/1.3.4.4.4.2.4: The vertex z is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.2.4: By way of contradiction, suppose z is adjacent to an additional vertex; call it q. See Figure 100(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $q \nsim s$ by birth. Since $q \nsim r$ by birth, $q \nsim t$ by claw-freedom at t. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex. ■

Therefore $r \nsim t$. ■

By a similar argument, we may also assume that $z \nsim s$. More generally, we may assume that neighbors (that are outside of a bow-tie) of any two distinct vertices of a bow-tie are independent.

Claim 2.4/1.3.4.4.4.3: The vertex x is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.3: By way of contradiction, suppose x is adjacent to an additional vertex; call it q. To prevent $\{xv, xz, xq\}$ from forming a claw at x, we must have $z \sim q$. See Figure 101(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing x, so that $V(C) = \{v, x, z, q\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that q is not adjacent to s, r or t by birth. Thus by Lemma 2.2, either x is adjacent to an additional vertex, or z or q is adjacent to an additional vertex (these two subcases are symmetric).

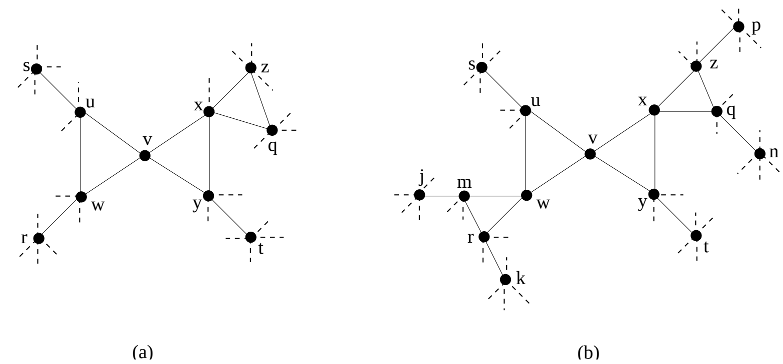


Figure 101: Proving that every vertex of degree four must lie on a K_4.

Suppose x is adjacent to an additional vertex; call it p. To prevent $\{xv, xz, xp\}$ and $\{xv, xq, xp\}$ from forming claws at x, we must have $z \sim p$ and $q \sim p$. Either p is interior to the xzq-face, q is interior to the xzp-face, or z is interior to the xqp-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r]$ containing x, so that $V(C) = \{v, x, z, q, p\}$. Then every vertex in $C - x$ is adjacent to x, vertices x, v and one of z, q and p cannot grow, and v is adjacent to none of z, q or p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex and so $d(x) = 4$.

Suppose, without loss of generality, that z is adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, r, p]$ containing x, so that $V(C) = \{v, x, q\}$. Then C is not well-covered, since $\{x\}$ and $\{v, q\}$ are both maximal independent sets of C. Note that p is not adjacent to s, r or t by birth. Also note that $p \nsim q$ by Claim 2.4/1.3.4.3, since z and q already share the neighbor x and x cannot grow. Thus by Lemma 2.2, q must be adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it m. To prevent $\{wv, wr, wm\}$ from forming a claw at w, we must have $r \sim m$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p]$ containing w, so that $V(C) = \{v, w, r, m\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that m is not adjacent to t, s, q or p by birth. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or or m is adjacent to an additional vertex (these two cases are symmetric).

Suppose w is adjacent to an additional vertex; call it k. To prevent $\{wv, wr, wk\}$ and $\{wv, wm, wk\}$ from forming claws at w, we must have $r \sim k$ and $m \sim k$. Either k is interior to the wrm-face, m is interior to the wrk-face, or r is interior to the wmk-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p]$ containing w, so that $V(C) = \{v, w, r, m, k\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of r, m and k cannot grow, and v is adjacent to none of r, m or k. Thus by Lemma 1.4, G is not well-covered, a contradiction. Thus w is not adjacent to a fifth vertex, and $d(w) = 4$.

Suppose r is adjacent to an additional vertex; call it k. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p, k]$ containing w, so that $V(C) = \{v, w, m\}$. Then C is not well-covered, since $\{w\}$ and $\{v, m\}$ are both maximal independent sets of C. Note that k is not adjacent to t, s, q, p or k by birth. Also note that $m \nsim k$ by Claim 2.4/1.3.4.3, since m and r already share the neighbor w and w cannot grow. Thus by Lemma 2.2, m must be adjacent to an additional vertex; call it j. See Figure 101(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, p, n, k, j]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that j is not adjacent to t, s, p, or k by birth. Also note that C cannot grow. Thus by Lemma 2.2, n is adjacent to either k or j (these two cases are symmetric). Suppose, without loss of generality, $n \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, s, n, j]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Since neither n nor j is adjacent to p by birth, neither n nor j is adjacent to z by claw-freedom at z. Note that $j \nsim r$ by Claim 2.4/1.3.4.3, since r and m already share the neighbor w and w cannot grow. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it i. To prevent $\{rw, rk, ri\}$ from forming a claw at r, we must have $k \sim i$. Call this semi-known subgraph S. Let C be the component of $S - N[t, n, s, p, j, i]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that i is not adjacent to t, n, s, or j by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $i \sim p$. See Figure 102(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, n, m, i]$ containing x, so that $V(C) = \{v, x, z\}$. Then C is not well-covered, since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Note that $m \nsim i$ by Claim 2.4/1.3.4.3, since m and r already share the neighbor w and w cannot grow. Since m is adjacent to neither q nor k, $m \nsim n$ by claw-freedom at n. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it h. To prevent $\{zx, zp, zh\}$ from forming a claw at z, we must have $p \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, n, j, i, h]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that h is not adjacent to t, s, n, and i by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $h \sim j$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, i, h]$ containing

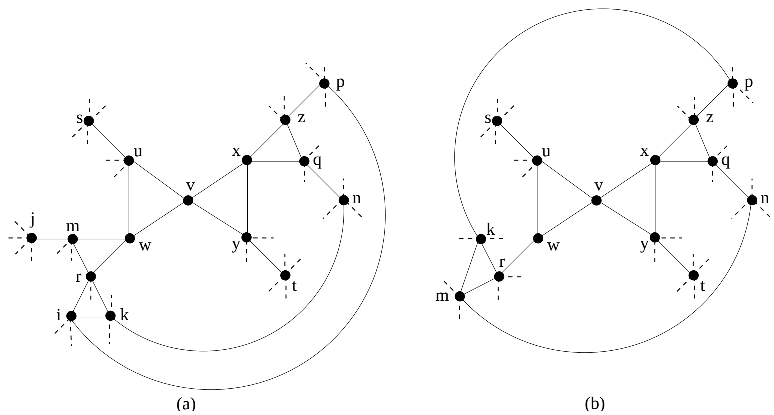


Figure 102: Proving that every vertex of degree four must lie on a K_4.

w, so that $V(C) = \{v, w, m\}$. Then C is not well-covered, since $\{w\}$ and $\{v, m\}$ are both maximal independent sets of C. Note that $q \nsim h$ by Claim 2.4/1.3.4.3, since z and q already share the neighbor x and x cannot grow. Since $i \nsim n$ by birth, $i \nsim q$ by claw-freedom at q. Note that $h \nsim m$ by birth. Thus by Lemma 2.2, m must be adjacent to an additional vertex; call it g. To prevent $\{mw, mj, mg\}$ from forming a claw at m, we must have $j \sim g$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, n, i, h, g]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that g is not adjacent to t, s, i or h by birth. Also note that $n \nsim g$ by planarity. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence neither k nor j is adjacent to n, and so neither r nor m is adjacent to an additional vertex. Therefore w is not adjacent to an additional vertex and so $d(w) = 3$.

Suppose r is adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, n, p, m]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that m is not adjacent to t, s, or p by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $m \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[z,t,s,n]$ containing w, so that $V(C) = \{v,w,r\}$. Then C is not well-covered, since $\{w\}$ and $\{v,r\}$ are both maximal independent sets of C. Since $n \nsim p$ by birth, $n \nsim z$ by claw-freedom at z. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it k. To prevent $\{rw, rm, rk\}$ from forming a claw at r, we must have $m \sim k$. Call this semi-known subgraph S. Let C be the component of $S - N[t,s,n,p,k]$ containing v, so that $V(C) = \{v,w,x\}$. Then C is not well-covered, since $\{v\}$ and $\{w,x\}$ are both maximal independent sets of C. Note that k is not adjacent to t, s or n by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $k \sim p$. See Figure 102(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y,z,s,n]$ containing r, so that $V(C) = \{w,r,k\}$. Then C is not well-covered, since $\{r\}$ and $\{w,k\}$ are both maximal independent sets of C. Since neither n nor k is adjacent to t by birth, neither n nor k is adjacent to y by claw-freedom at y. Thus by Lemma 2.2, either r is adjacent to an additional vertex, or k is adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it j. To prevent $\{rw, rm, rj\}$ and $\{rw, rk, rj\}$ from forming claws at r, we must have $m \sim j$ and $k \sim j$. Thus by planarity, j must be interior to the rmk-face. Note that r cannot have any additional neighbors in the $rkpzxvuw$-face; otherwise this neighbor together with w and m would form a claw contradicting the fact that G is claw-free. Also r cannot have any additional neighbors in the rkj, rmj, or $rmnqxyvw$ faces; otherwise such an additional vertex together with w and m, k, or j respectively, would produce a claw at r. Thus r cannot grow and $d(r) = 4$. Hence by Claim 2.4/1.2, j cannot grow and $d(j) = 3$. Call this semi-known subgraph S. Let C be the component of $S - N[y,z,s,n]$ containing r, so that $V(C) = \{w,r,k,j\}$. Then every vertex of $C - r$ is adjacent to r, vertices r, w and j cannot grow, and $w \nsim j$. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence r is not adjacent to an additional vertex, and $d(r) = 3$.

Suppose k is adjacent to an additional vertex; call it j. To prevent $\{kr, kp, kj\}$ from forming a claw at k, we must have $j \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, s, n, j]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that j is not adjacent to z, s or n by birth, and C cannot grow. Thus by Lemma 2.2, we must have $j \sim t$. See Figure 103(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, s, q, j]$ containing r, so that $V(C) = \{w, r, m\}$. Then C is not well-covered, since $\{r\}$ and $\{w, m\}$ are both maximal independent sets of C. Since $j \nsim n$ by birth, $j \nsim q$ by claw-freedom at q. Since neither q nor m are adjacent to t by birth, neither q nor m is adjacent to y by claw-freedom at y. Thus by Lemma 2.2, m must be adjacent to an additional vertex; call it i. To prevent $\{mr, mn, mi\}$ from forming a claw at m, we must have $n \sim i$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p, i]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that i is not adjacent to either s or q by birth, and $i \nsim t$ by planarity. Also note that C cannot grow. Thus by Lemma 2.2, we must have $i \sim p$. To prevent $\{pz, pj, pi\}$ from forming a claw at p, we must have $i \sim z$, since $j \nsim z$ by birth, and $i \nsim j$ by planarity. Call this semi-known subgraph S. Let C be the component of $S - N[x, t, s, i]$ containing r, so that $V(C) = \{w, r, k\}$. Then C is not well-covered, since $\{r\}$ and $\{w, k\}$ are both maximal independent sets of C. Since $i \nsim j$ by birth, $i \nsim k$ by claw-freedom at k. Thus by Lemma 2.2, k must be adjacent to an additional vertex; call it h. To prevent $\{kr, kp, kh\}$ and $\{kr, kj, kh\}$ from forming claws at k, we must have $p \sim h$ and $j \sim h$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, i, h]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that h is not adjacent to t, s or i by birth, and

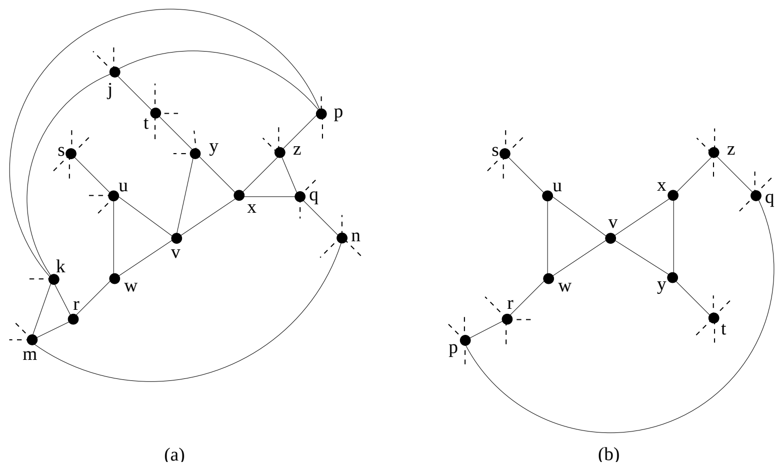


Figure 103: Proving that every vertex of degree four must lie on a K_4.

$h \nsim q$ by planarity. Also note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence k is not adjacent to an additional vertex and so r is not adjacent to an additional vertex.

Therefore neither z nor q is adjacent to an additional vertex and so x is not adjacent to an additional vertex and so $d(x) = 3$. ■

By a similar arguments, we may also assume that all of u, w and y also have degree three and cannot grow.

Claim 2.4/1.3.4.4.4.4: The vertex z is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.4.4.4.4: By way of contradiction, suppose z is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, s]$ containing w, so that $V(C) = \{v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{v, r\}$ are both maximal independent sets of C. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, p]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that neither q nor p are adjacent to either t or

s by their births. Also note that C cannot grow. Thus by Lemma 2.2, we must have $p \sim q$. See Figure 103(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, r]$ containing u, so that $V(C) = \{u, v, s\}$. Then C is not well-covered, since $\{u\}$ and $\{v, s\}$ are both maximal independent sets of C. Thus by Lemma 2.2, s must be adjacent to an additional vertex; call it n. Call this semi-known subgraph S. Let C be the component of $S - N[t, r, q, n]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered, since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Note that n is not adjacent to r or t by birth. Also note that C cannot grow. Thus by Lemma 2.2, we must have $n \sim q$. To prevent $\{qz, qp, qn\}$ from forming a claw at q, we must have $p \sim n$, since neither p nor n is adjacent to z by birth. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, n]$ containing w, so that $V(C) = \{u, v, w, r\}$. Then C is not well-covered, since $\{w\}$ and $\{u, r\}$ are both maximal independent sets of C. Recall that n is adjacent to neither z nor t by birth. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it m. To prevent $\{rw, rp, rm\}$ from forming a claw at r, we must have $m \sim p$. Call this semi-known subgraph S. Let C be the component of $S - N[t, s, q, m]$ containing v, so that $V(C) = \{v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{w, x\}$ are both maximal independent sets of C. Note that $m \nsim t$ by birth. Since $m \nsim n$ by birth, $m \nsim s$ by claw-freedom at s. Also note that $m \nsim t$ otherwise we would have a forbidden subgraph centered at p by Claim 2.4/1.3.4.1. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex. ■

Therefore y is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.4.4.4. ■

But if $y \nsim z$ by Claim 2.4/1.3.4.4.1, $y \nsim w$ by Claim 2.4/1.3.4.1, $y \nsim u$ by Claim 2.4/1.3.4.2, and y is not adjacent to an additional vertex by Claim 2.4/1.3.4.4.4, then $d(y) = 2$, contradicting the fact that G is 3-connected. Thus we do not need to even consider whether u or w is adjacent to an additional vertex. Therefore x is not adjacent to an additional vertex, and we have prove Claim 2.4/1.3.4.4. ■

By Claim 2.4/1.3.4.1, $x \nsim u$, by Claim 2.4/1.3.4.2, $x \nsim w$, and by Claim 2.4/1.3.4.4 x is not adjacent to an additional vertex. But then $d(x) = 2$, which contradicts the fact that G is 3-connected. Hence if G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 4$, then v must lie on a K_4. We have proved Claim 2.4/1.3.4. ■

Claim 2.4/1.3.5: If G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 3$, then v must lie on a K_4.

Proof of Claim 2.4/1.3.5: Let G be a graph and v be a vertex of G that fulfill the hypotheses of the claim. Label the neighbors of v in a clockwise fashion: u, w, x. Suppose, by way of contradiction, that v does not lie on a K_4. By claw-freedom, and without loss of generality, we may assume that $w \sim x$. Since G is 3-connected, every vertex of G has degree at least three. Thus w must grow. Either w is adjacent to u, or w is adjacent to an additional vertex.

Suppose $w \sim u$. Then since v does not lie on a K_4, $u \nsim x$. Suppose w is adjacent to an additional vertex; call it y. See Figure 104(a) for an illustration. To prevent $\{wu, wx, wy\}$ from forming a claw at w, either $u \sim y$ or $x \sim y$. In either case, we obtain a forbidden subgraph centered at w, by Claim 2.4/1.3.4.1. Thus w cannot be adjacent to an additional vertex and so $d(w) = 3$. But then $\{u, x\}$ is a 2-cut, separating v and w from the rest of the graph and contradicting the fact that G is 3-connected. Therefore $w \nsim u$, and by symmetry we may assume that $u \nsim x$.

Suppose w is adjacent to an additional vertex; call it y. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing v, so that $V(C) = \{u, v, x\}$.

Figure 104: Proving that every vertex of degree three must lie on a K_4.

Then C is not well-covered, since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Note that $x \nsim y$ by Claim 2.4/1.3.4.3, since w and x already share the neighbor v, and v cannot grow. Thus by Lemma 2.2, either u is adjacent to y, x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Claim 2.4/1.3.5.1: The vertex u is not adjacent to y.

Proof of Claim 2.4/1.3.5.1: Suppose, by way of contradiction, that $u \sim y$. Since G is 3-connected, $\{v, w\}$ is not a 2-cut, and so there must be a path from x to y that does not pass through either v or w. Since $x \nsim y$, x must be adjacent to an additional vertex; call it z. See Figure 104(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[u]$ containing x, so that $V(C) = \{w, x, z\}$. Then C is not well-covered, since $\{x\}$ and $\{w, z\}$ are both maximal independent sets of C. Note that $w \nsim z$ by Claim 2.4/1.3.4.3, since w and x already share the neighbor v, and v cannot grow. Thus by Lemma 2.2, either u is adjacent to z, w is adjacent to an additional vertex, x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Claim 2.4/1.3.5.1.1: The vertex u is not adjacent to z.

Proof of Claim 2.4/1.3.5.1.1: Suppose, by way of contradiction, that $u \sim z$. To prevent $\{uv, uy, uz\}$ from forming a claw at u, we must have $u \sim z$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(l). Since G is not a graph from the Figure 2, this graph must grow. There are no possible additional edges between known vertices, so there must exist additional vertices. Either w or x is adjacent to an additional vertex (these two cases are symmetric), u is adjacent to an additional vertex, y is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Suppose, without loss of generality, that w is adjacent to an additional vertex; call it t. To prevent $\{wv, wy, wt\}$ from forming a claw at w, we must have $y \sim t$. Note that t could be in either the $wvuy$-face or the $wxzy$-face. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing w, so that $V(C) = \{v, w, t\}$. Then C is not well-covered, since $\{w\}$ and $\{v, t\}$ are both maximal independent sets of C. Note that $t \nsim z$ by Claim 2.4/1.3.4.3, since t and y already share the neighbor w, and $w \nsim z$. Thus by Lemma 2.2, either w is adjacent to an additional vertex, or t is adjacent to an additional vertex. Suppose w is adjacent to an additional vertex; call it s. To prevent $\{wv, wy, ws\}$ and $\{wv, wt, ws\}$ from forming claws at w, we must have $y \sim s$ and $t \sim s$. Either s is interior to the wyt-face or t is interior to the wys-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, whichever of s or t is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[z]$ containing w, so that $V(C) = \{v, w, t, s\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v and one of t and s cannot grow, and v is adjacent to neither t nor s. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to a fifth additional vertex and so $d(w) = 4$. Suppose t is adjacent to an additional vertex; call it s. Since $d(w) = 4$, w must lie on a K_4 by Claim 2.4/1.3.4. But v cannot grow, and neither y nor t is adjacent to x by Claim 2.4/1.3.4.3, and so w does not lie on a K_4, a contradiction. Hence t is not adjacent to an additional vertex and so neither w nor x is adjacent to an additional vertex.

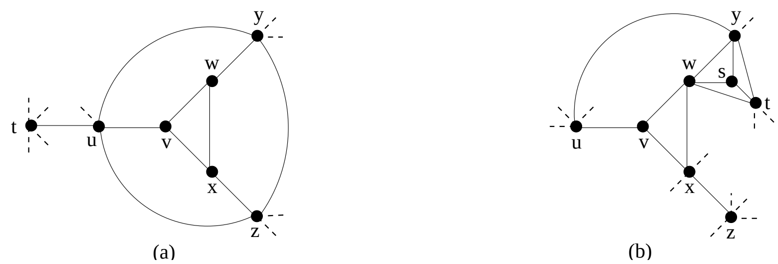


Figure 105: Proving that every vertex of degree three must lie on a K_4.

We may now assume $d(w) = d(x) = 3$. Note that since G is 3-connected and these vertices cannot grow, there are no additional vertices in either the $wxzy$-face, the $vxzu$-face, or the $vuyw$-face.

Suppose u is adjacent to an additional vertex; call it t. See Figure 105(a) for an illustration. This vertex must be in the uyz-face. To prevent $\{uv, uy, ut\}$ and $\{uv, uz, ut\}$ from forming claws at u, we must have $y \sim t$ and $z \sim t$. Then this graph is isomorphic to the exceptional well-covered graph shown in Figure 2(k). Since G is not a graph from the Figure 2, this graph must grow. Note that y cannot be adjacent to any additional vertices in the uyt-face; otherwise the additional neighbor together with w and z would form a claw at y, contradicting the fact that G is claw-free. Thus by 3-connectedness, there are no additional neighbors in the uyt-face. Also note that z cannot be adjacent to any additional vertices in the uzt-face; otherwise the additional neighbor together with x and y would form a claw at z, contradicting the fact that G is claw-free. Thus by 3-connectedness, there are no additional neighbors in the uzt-face. Finally, note that y cannot be adjacent to any additional vertices in the yzt-face; otherwise the additional neighbor together with w and u would form a claw at y, contradicting the fact that G is claw-free. Thus by 3-connectedness, there are no additional neighbors in the yzt-face. But then the graph cannot grow, a contradiction. Hence u is not adjacent to an additional vertex.

Suppose y is adjacent to an additional vertex; call it t. But then $\{yw, yu, yt\}$ is a claw at y, since u and w cannot grow. This contradicts that fact that G is claw-free, and so y is not adjacent to an additional vertex.

Suppose z is adjacent to an additional vertex; call it t. But then $\{zu, zx, zt\}$ is a claw at z, since u and x cannot grow. This contradicts that fact that G is claw-free, and so z is not adjacent to an additional vertex.

Therefore $u \nsim z$ and we have proved Claim 2.4/1.3.5.1.1. ■

Note that by symmetry we may also assume that $y \nsim z$.

Claim 2.4/1.3.5.1.2: The vertex w is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.1.2: Suppose, by way of contradiction, that w is adjacent to an additional vertex; call it t. To prevent $\{wv, wy, wt\}$ from forming a claw at w, we must have $y \sim t$. Note that t is in either the $uvwy$-face or the exterior face. Since $d(w) \geq 4$, w must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that $t \nsim x$ by Claim 2.4/1.3.4.3, since w and x already share the neighbor v and v cannot grow. Thus w must be adjacent to a fifth neighbor, call it s, to form a K_4 with y and t (i.e. $s \sim y$ and $s \sim t$). Either s is interior to the wyt-face, or t is interior to the wys-face. Now $d(w) = 5$, and so by Claim 2.4/1.2, whichever of s or t is interior cannot grow and must have degree three. See Figure 105(b) for a possible illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing x, so that $V(C) = \{v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either x is adjacent to an additional vertex, or z is adjacent to an additional vertex.

Suppose that x is adjacent to an additional vertex; call it r. To prevent $\{xv, xz, xr\}$ from forming a claw at x, we must have $z \sim r$. Since $d(x) \geq 4$, x must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v and w cannot grow. Thus x must be adjacent to a fifth neighbor, call it q, to form a K_4 with z and r (i.e. $q \sim z$ and $q \sim r$). Either q is interior to the xzr-face, r is interior to the xzq-face, or z is interior to the xrq-face. Now $d(x) = 5$, and so by Claim 2.4/1.2, whichever of q, r or z is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y]$ containing x, so that $V(C) = \{v, x, z, r, q\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, v and one of z,r and q cannot grow, and v is adjacent to none of z, r or q. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex and so $d(x) = 3$.

Suppose z is adjacent to an additional vertex; call it r. Call this semi-known subgraph S. Let C be the component of $S - N[u, r]$ containing w, so that $V(C) = \{x, w, t, s\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, x and one of t and s cannot grow, and x is adjacent to neither t nor s. Since $r \nsim y$ by birth, $r \nsim u$ by claw-freedom at u. Thus by Lemma 2.3, G is not well-covered, a contradiction. Thus z is not adjacent to an additional vertex, and therefore w is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.5.1.2. ■

Thus we may assume that $d(w) = 3$.

Claim 2.4/1.3.5.1.3: The vertex x is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.1.3: Suppose, by way of contradiction, that x is adjacent to an additional vertex; call it t. To prevent $\{xv, xz, xt\}$ from forming a claw at x, we must have $z \sim t$. Since $d(x) \geq 4$, x must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v and w cannot grow. Thus x must be adjacent to a fifth neighbor, call it s, to form a K_4 with z and t (i.e. $s \sim z$ and $s \sim t$). Either s is interior to the xzt-face, t is interior to the xzs-face, or z is interior to the xts-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[u]$ containing x, so that $V(C) = \{w, x, z, t, s\}$. Then every vertex of $C - x$

is adjacent to x, vertices x, w, and one of z, t, and s cannot grow, and w is adjacent to none of z, t, and s. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex. ■

Claim 2.4/1.3.5.1.4: The vertex z is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.1.4: Suppose, by way of contradiction, that z is adjacent to an additional vertex; call it t. Since G is 3-connected, $\{u, z\}$ is not a 2-cut, and so there must exist a path from y to t that does not pass through either u or z. Since $t \nsim u$ by birth, $t \nsim y$ by claw-freedom at y. Thus y must be adjacent to an additional vertex; call it s. To prevent $\{yw, yu, ys\}$ from forming a claw at y, we must have $u \sim s$. Since G is 3-connected, $\{z, s\}$ is not a 2-cut, and so there must exist a path from u and y to t that does not pass through either z or s. Thus and by claw-freedom, u and y must share an additional neighbor, call it r. Since r is a vertex on a path from u and y to t that does not pass though s, r must be in the exterior face. To prevent $\{yw, ys, yr\}$ from forming a claw at y, we must have $s \sim r$. Note that y cannot be adjacent to an additional vertex in the exterior face; otherwise this additional vertex together with w and s would form a claw at y, contradicting the fact that G is claw-free. Also note that u cannot be adjacent to an additional vertex in the exterior face; otherwise this additional vertex together with v and s would form a claw at u, contradicting the fact that G is claw-free. But then $\{z, r\}$ is a 2-cut, separating t from the rest of the graph, and contradicting the fact that G is 3-connected. Hence z is not adjacent to an additional vertex. ■

Therefore $u \nsim y$, and we have proved Claim 2.4/1.3.5.1. ■

Claim 2.4/1.3.5.2: The vertex x is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.2: Suppose, by way of contradiction, that x is adjacent to an additional vertex; call it z. Since G is 3-connected, v is not a cut-vertex and

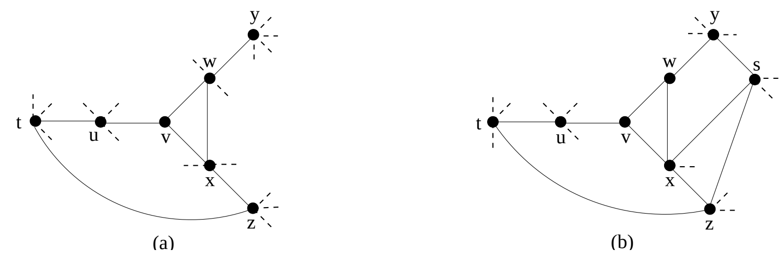


Figure 106: Proving that every vertex of degree three must lie on a K_4.

so there must be a path from the set $\{w, x, y, z\}$ to u that does not pass through v. Note that by symmetry with Claim 2.4/1.3.5.1, we may assume that $z \nsim u$. Thus u must be adjacent to an additional vertex; call it t. Call this semi-known subgraph S. Let C be the component of $S - N[z,t]$ containing w, so that $V(C) = \{v, w, y\}$. Then C is not well-covered, since $\{w\}$ and $\{v, y\}$ are both maximal independent sets of C. Note that $z \nsim y$ by birth. Also note that $z \nsim w$ by Claim 2.4/1.3.4.3, since x and w already share the neighbor v and v cannot grow. Thus by Lemma 2.2, either t is adjacent to z or y (the cases for $t \sim z$ and $t \sim y$ are symmetric), t is adjacent to w, vertices w is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Claim 2.4/1.3.5.2.1: The vertex t is not adjacent to z.

Proof of Claim 2.4/1.3.5.2.1: Suppose, by way of contradiction, that $t \sim z$. See Figure 106(a) for an illustration. Note that $t \nsim y$; otherwise there would be a claw at t, since $z \nsim y$ by birth and neither y nor z is adjacent to u by Claim 2.4/1.3.5.1. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing w, so that $V(C) = \{v, w, x, y\}$. Then every vertex of $C - w$ is adjacent to w, v cannot grow, and $v \nsim y$. If w and y cannot grow, then by Lemma 2.3, G is not well-covered, a contradiction. Thus either w is adjacent to an additional vertex, or y is adjacent to an additional vertex.

Suppose w is adjacent to an additional vertex; call it s. To prevent $\{wv, wy, ws\}$ from forming a claw at w, we must have $y \sim s$. Since $d(w) \geq 4$, w must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v cannot grow, and y and s are not adjacent to x by Claim 2.4/1.3.4.3. Thus w must be adjacent to a fifth neighbor, call it r, to form a K_4 with y and s (i.e. $r \sim y$ and $r \sim s$). Either r is interior to the wys-face, s is interior to the wyr-face, or y is interior to the wsr-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[t]$ containing w, so that $V(C) = \{v, w, x, y, s, r\}$. Then every vertex of $C - w$ is adjacent to w, vertices w, v, and one of y, s, and r cannot grow, and v is adjacent to none of y, s, and r. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex, and $d(w) = 3$.

Suppose y is adjacent to an additional vertex; call it s. Call this semi-known subgraph S. Let C be the component of $S - N[u, s]$ containing x, so that $V(C) = \{w, x, z\}$. Then C is not well-covered, since $\{x\}$ and $\{w, z\}$ are both maximal independent sets of C. Since $s \nsim t$ by birth, $s \nsim u$ by claw-freedom at u. Recall that u is not adjacent to x (by a previous subcase among the subcases of Claim 2.4/1.3.5) or z (by symmetry with Claim 2.4/1.3.5.1). Note that if $s \sim x$, then we must have $s \sim z$ by claw-freedom at z. Also if $s \sim z$, then we must have $s \sim x$ by claw-freedom at z since $s \nsim t$ by birth. Thus by Lemma 2.2, either s is adjacent to both z and x, vertices x is adjacent to an additional vertex, or z is adjacent to an additional vertex. Suppose s is adjacent to both z and x. See Figure 106(b) for an illustration. Now since $d(x) \geq 4$, x must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v and w cannot grow. Thus x must be adjacent to a fifth neighbor, call it r, to form a K_4 with z and s (i.e. $r \sim z$ and $r \sim s$). Thus r must be interior to the xzs-face by planarity. Now $d(x) = 5$ and so by Claim 2.4/1.2, r cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of

$S - N[y,t]$ containing x, so that $V(C) = \{v, x, r\}$. Then C is not well-covered, since $\{x\}$ and $\{v, r\}$ are both maximal independent sets of C. Recall that $t \nsim y$ by claw-freedom, and note that C cannot grow. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence s is adjacent to neither z nor x.

Suppose x is adjacent to an additional vertex; call it r. To prevent $\{xv, xz, xr\}$ from forming a claw at x, we must have $z \sim r$. Now since $d(x) \geq 4$, x must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v and w cannot grow. Thus x must be adjacent to a fifth neighbor, call it q, to form a K_4 with z and r (i.e. $q \sim z$ and $q \sim r$). Either q is interior to the xzr-face, or r is interior to the xzq-face. Now $d(x) = 5$ and so by Claim 2.4/1.2, whichever of q or r is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[u, s]$ containing x, so that $V(C) = \{w, x, z, r, q\}$. Then every vertex of $C - x$ is adjacent to x, vertices x, w and one of z, r and q cannot grow, and w is adjacent to none of z, r or q. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence x is not adjacent to an additional vertex and so $d(x) = 3$.

Suppose z is adjacent to an additional vertex; call it r. To prevent $\{zx, zt, zr\}$ from forming a claw at z, we must have $t \sim r$. See Figure 107(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[s, r]$ containing v, so that $V(C) = \{u, v, w, x\}$. Then C is not well-covered, since $\{v\}$ and $\{u, w\}$ are both maximal independent sets of C. Note that r is adjacent to neither u nor s by birth. Thus by Lemma 2.2, u must be adjacent to an additional vertex; call it q. To prevent $\{uv, ut, uq\}$ from forming a claw at u, we must have $t \sim q$. Now since $d(t) \geq 4$, t must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that there are no edges between the sets $\{u, q\}$ and $\{z, r\}$. Thus t must be adjacent to an additional vertex; call it p, which, together with t, is either in a K_4 with u and q, or in a K_4 with z and r. Since these cases are symmetric, assume without loss of generality

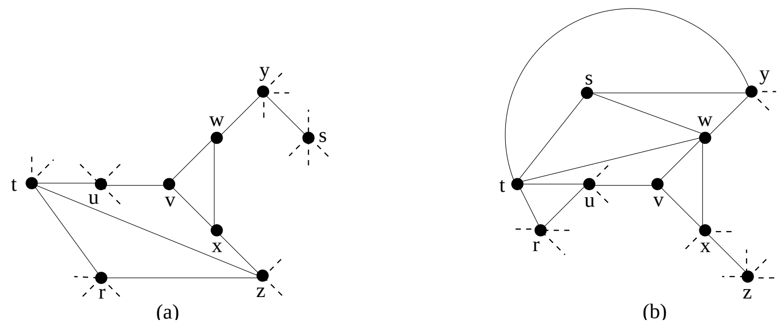


Figure 107: Proving that every vertex of degree three must lie on a K_4.

that t and p are in a K_4 with u and q (i.e. p is adjacent to u, t, and q). Either p is interior to the utq-face, or q is interior to the utp-face. Now $d(t) = 5$ and so by Claim 2.4/1.2, whichever of p or q is interior cannot grow and must have degree three. Also note that $d(u) = 4$ and u cannot grow; otherwise an additional neighbor together with v and p would form a claw at u, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[y, z]$ containing u, so that $V(C) = \{u, v, q, p\}$. Then every vertex of $C - u$ is adjacent to u, vertices u, v and one of q and p cannot grow, and v is adjacent to neither q nor p. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence z is not adjacent to an additional vertex, y is not adjacent to an additional vertex.

Therefore $t \nsim z$ and we have proved Claim 2.4/1.3.5.2.1. ■

Note that by symmetry, we may also now assume that $t \nsim y$.

Claim 2.4/1.3.5.2.2: The vertex t is not adjacent to w.

Proof of Claim 2.4/1.3.5.2.2: Suppose, by way of contradiction, that $t \sim w$. To prevent $\{wv, wy, wt\}$ from forming a claw at w, we must have $t \sim y$. Now since $d(w) \geq 4$, w must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v cannot grow, $x \nsim y$ by Claim 2.4/1.3.4.3, and since $t \nsim z$, by birth $t \nsim x$ by claw-freedom at x. Thus w must be adjacent to a fifth neighbor, call it s, to form a K_4 with y

and t (i.e. $s \sim y$ and $s \sim t$). Either s is interior to the wyt-face, or y is interior to the wts-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever of s or y is interior cannot grow and must have degree three. Since G is 3-connected, $\{v, t\}$ is not a 2-cut, and so there must be a path from u to the set $\{x, y, z, s\}$ that does not pass through either v or t. Note that $u \nsim s$ by Claim 2.4/1.3.4.3, since t and s already share the neighbor w, and w cannot grow. Thus u must be adjacent to an additional vertex; call it r. To prevent $\{uv, ut, ur\}$ from forming a claw at u, we must have $r \sim t$. See Figure 107(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[z, r]$ containing w, so that $V(C) = \{v, w, y, s\}$. Then every vertex of $C - w$ is adjacent to w, the vertices w, v and one of y and s cannot grow, and v is adjacent to neither y nor s. Thus if $\{z, r\}$ is an independent set, then by Lemma 2.3, G is not well-covered, a contradiction. Hence we must have $z \sim r$. Call this semi-known subgraph S. Let C be the component of $S - N[r]$ containing w, so that $V(C) = \{v, w, x, y, s\}$. Then every vertex of $C - w$ is adjacent to w, the vertices w, v and one of y and s cannot grow, and v is adjacent to neither y nor s. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence $t \nsim w$. ■

Claim 2.4/1.3.5.2.3: The vertex w is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.2.3: Suppose, by way of contradiction, that w is adjacent to an additional vertex; call it s. To prevent $\{wv, wy, ws\}$ from forming a claw at w, we must have $y \sim s$. Now since $d(w) \geq 4$, w must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that v cannot grow, and x is adjacent to neither y nor s by Claim 2.4/1.3.4.3. Thus w must be adjacent to a fifth neighbor, call it r, to form a K_4 with y and s (i.e. $r \sim y$ and $r \sim s$). Either r is interior to the wys-face, s is interior to the wyr-face, or y is interior to the wsr-face. Now $d(w) = 5$ and so by Claim 2.4/1.2, whichever vertex is interior cannot grow and must have degree three. See Figure 108(a) for a possible illustration. Call this semi-known subgraph S. Let

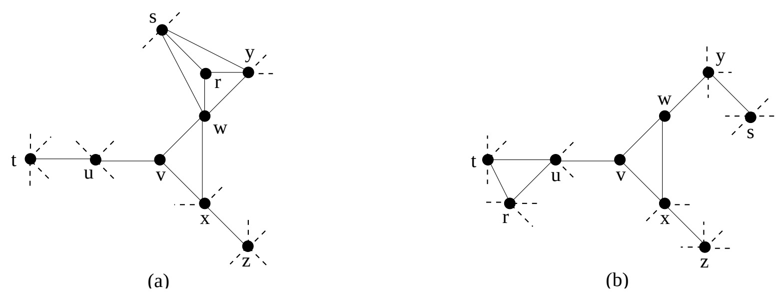


Figure 108: Proving that every vertex of degree three must lie on a K_4.

C be the component of $S - N[z, t]$ containing w, so that $V(C) = \{v, w, y, s, r\}$. Then every vertex of $C - w$ is adjacent to w, the vertices w, v and one of y, s, and r cannot grow, and v is adjacent to none of y, s, or r. Recall that $t \nsim z$ by Claim 2.4/1.3.5.2.1. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence w is not adjacent to an additional vertex. ■

We may now assume that $d(w) = 3$. By symmetry of the graph, this also implies that $d(x) = 3$ as well.

Claim 2.4/1.3.5.2.4: The vertex y is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.2.4: Suppose, by way of contradiction, that y is adjacent to an additional vertex; call it s. Call this semi-known subgraph S. Let C be the component of $S - N[y, z]$ containing u, so that $V(C) = \{u, v, t\}$. Then C is not well-covered since $\{u\}$ and $\{v, t\}$ are both maximal independent sets of C. Recall that $z \nsim y$ by birth. Thus by Lemma 2.2, either u is adjacent to an additional vertex, or t is adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it r. To prevent $\{uv, ut, ur\}$ from forming a claw at u, we must have $t \sim r$. See Figure 108(b) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[y, z]$ containing u, so that $V(C) = \{u, v, t, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, t\}$ are both maximal independent sets of C. Thus by Lemma 2.2, either u is adjacent to an additional vertex, t is adjacent to an additional vertex, or r is adjacent to an additional vertex.

Suppose u is adjacent to an additional vertex; call it q. To prevent $\{uv, ut, uq\}$ and $\{uv, ur, uq\}$ from forming claws at u, we must have $t \sim q$ and $r \sim q$. Either q is interior to the utr-face, r is interior to the utq-face, or t is interior to the urq-face. Suppose q is interior to the utr-face. Note then that u cannot have any additional neighbors in the exterior face; otherwise this neighbor together with v and q would form a claw contradicting the fact that G is claw-free. Also u cannot have any additional neighbors in the urq, or utq faces; otherwise such an additional vertex together with v and t, or r respectively, would produce a claw at u. We can make similar arguments if r is interior to the utq-face or t is interior to the urq-face. Thus u cannot grow and $d(u) = 4$. Hence by Claim 2.4/1.2, whichever of q, r or t is interior cannot grow and must have degree three. Call this semi-known subgraph S. Let C be the component of $S - N[y, z]$ containing u, so that $V(C) = \{u, v, t, r, q\}$. Then every vertex of $C - u$ is adjacent to u, the vertices u, v, and one of t, r and q cannot grow, and v is adjacent to none of t, r or q. Thus by Lemma 2.3, G is not well-covered, a contradiction. Hence u is not adjacent to a fifth additional vertex and so $d(u) = 4$.

Suppose t is adjacent to an additional vertex; call it q. Call this semi-known subgraph S. Let C be the component of $S - N[y, z, q]$ containing u, so that $V(C) = \{u, v, r\}$. Then C is not well-covered since $\{u\}$ and $\{v, r\}$ are both maximal independent sets of C. Note that q is adjacent to neither y nor z by birth. Also note that $q \nsim r$ by Claim 2.4/1.3.4.3, since r and t already share the neighbor u and u cannot grow. Thus by Lemma 2.2, r must be adjacent to an additional vertex; call it p. Call this semi-known subgraph S. Let C be the component of $S - N[z, s, q, p]$ containing

v, so that $V(C) = \{u, v, w\}$. Then C is not well-covered since $\{v\}$ and $\{u, w\}$ are both maximal independent sets of C. Note that p is adjacent to neither s nor q by birth, and C cannot grow. Thus by Lemma 2.2, p or q is adjacent to s (note that these two subcases are symmetric, so we need only consider one). Suppose, without loss of generality, that $p \sim s$. Call this semi-known subgraph S. Let C be the component of $S - N[z, t, p]$ containing w, so that $V(C) = \{v, w, y\}$. Then C is not well-covered since $\{w\}$ and $\{v, y\}$ are both maximal independent sets of C. Note that $p \nsim t$ by Claim 2.4/1.3.4.3, since r and t already share the neighbor u and u cannot grow. Recall that $p \nsim y$ by birth. Thus by Lemma 2.2, y must be adjacent to an additional vertex; call it n. To prevent $\{yw, ys, yn\}$ from forming a claw at y, we must have $s \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[z, q, p, n]$ containing v, so that $V(C) = \{u, v, w\}$. Then C is not well-covered since $\{v\}$ and $\{u, w\}$ are both maximal independent sets of C. Note that n is adjacent to neither z nor p by birth, and C cannot grow. Thus by Lemma 2.2, we must have $q \sim n$. Call this semi-known subgraph S. Let C be the component of $S - N[y, t, r]$ containing x, so that $V(C) = \{v, x, z\}$. Then C is not well-covered since $\{x\}$ and $\{v, z\}$ are both maximal independent sets of C. Thus by Lemma 2.2, z must be adjacent to an additional vertex; call it m. Call this semi-known subgraph S. Let C be the component of $S - N[y, q, p, m]$ containing v, so that $V(C) = \{u, v, x\}$. Then C is not well-covered since $\{v\}$ and $\{u, x\}$ are both maximal independent sets of C. Note that m is adjacent to neither y or q by birth, and C cannot grow. Thus by Lemma 2.2, we must have $p \sim m$. To prevent $\{pr, ps, pm\}$ from forming a claw at p, we must have $s \sim m$ since $m \nsim r$ by birth, and since $s \nsim t$ by birth we may assume that $s \nsim r$ by symmetry. Now since $d(s) \geq 4$, s must lie on a K_4 by Claims 2.4/1.3.3 and 2.4/1.3.4. Note that there are no edges between the sets $\{y, n\}$ and $\{m, p\}$. Thus s must be adjacent to a fifth neighbor, call it k, to form a K_4 with either y and n or m and p. Suppose k forms a K_4 with y and n (i.e. $k \sim y$ and $k \sim n$). By planarity,

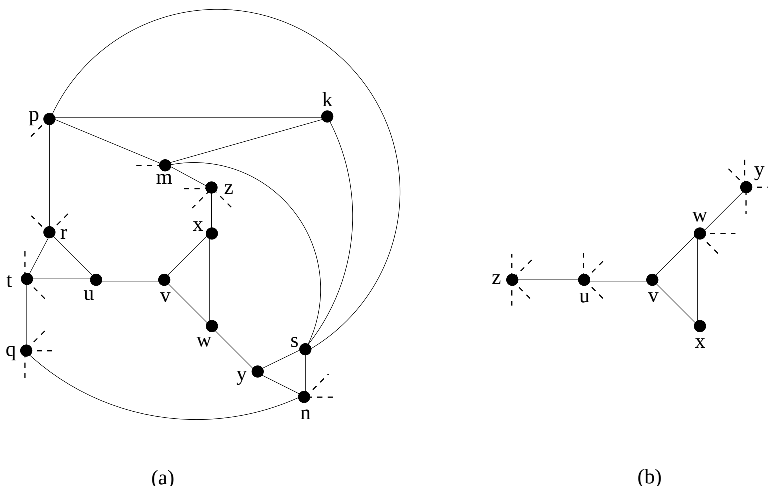


Figure 109: Proving that every vertex of degree three must lie on a K_4.

k must be interior to the ysn-face. Now $d(s) = 5$ and so by Claim 2.4/1.2, k cannot degree and so $d(k) = 3$. Also note that y cannot grow since if y were adjacent to an additional vertex, this vertex, together with w and k would form a claw at y, contradicting the fact that G is claw-free. Call this semi-known subgraph S. Let C be the component of $S - N[u, z, q, p]$ containing y, so that $V(C) = \{w, y, k\}$. Then C is not well-covered since $\{y\}$ and $\{w, k\}$ are both maximal independent sets of C. Recall that z is not adjacent to either p nor q by their births. Thus by Lemma 1.4, G is not well-covered, a contradiction. Suppose k forms a K_4 with p and m (i.e. $k \sim p$ and $k \sim m$). By planarity, k must be interior to the ypm-face. Now $d(s) = 5$ and so by Claim 2.4/1.2, k cannot degree and so $d(k) = 3$. Also note that y cannot grow since if y were adjacent to an additional vertex, this vertex, together with w and k would form a claw at y, contradicting the fact that G is claw-free. See Figure 109(a) for an illustration. Call this semi-known subgraph S. Let C be the component of $S - N[v, z, r, q]$ containing s, so that $V(C) = \{y, s, k\}$. Then C is not well-covered since $\{s\}$ and $\{y, k\}$ are both maximal independent sets of C. Recall that z is not adjacent to either r nor q by their births. Thus by Lemma 1.4, G is not well-covered, a contradiction. Hence t is not adjacent to an additional vertex.

Suppose r is adjacent to an additional vertex; call it q. Recall that by Claim 2.4/1.3.5.2.1, $t \nsim z$. By symmetry, this also implies that $t \nsim y$. Also recall that $t \nsim s$ by birth. Thus since t is not adjacent to an additional vertex (and this includes q) by the preceding subcase, $d(t) = 2$, contradicting the fact that G is 3-connected. Hence r is not adjacent to an additional vertex.

Therefore u is not adjacent to an additional vertex.

Suppose t is adjacent to an additional vertex; call it r. Since $s \nsim t$ by birth, $s \nsim u$ by claw-freedom at u. Recall that by Claim 2.4/1.3.5.1, $u \nsim y$. By symmetry, this also implies that $u \nsim z$. Thus since u is not adjacent to an additional vertex (and this includes r) by the preceding subcase, $d(u) = 2$, contradicting the fact that G is 3-connected. Hence t is not adjacent to an additional vertex.

Therefore y is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.5.2.4. ■

Hence x is not adjacent to an additional vertex, and we have proved Claim 2.4/1.3.5.2. ■

Claim 2.4/1.3.5.3: The vertex u is not adjacent to an additional vertex.

Proof of Claim 2.4/1.3.5.3: Suppose, by way of contradiction, that u is adjacent to an additional vertex; call it z. See Figure 109(b) for an illustration. Recall that $u \nsim x$ by symmetry since $u \nsim w$. Also recall that $x \nsim y$ by Claim 2.4/1.3.4.3. Thus since x is not adjacent to an additional vertex (and this includes z) by Claim 2.4/1.3.5.2, $d(x) = 2$, contradicting the fact that G is claw-free. Hence u is not adjacent to an additional vertex. ■

By Claim 2.4/1.3.5.1, $u \nsim y$. By Claim 2.4/1.3.5.2, x is not adjacent to an additional vertex. By Claim 2.4/1.3.5.3, u is not adjacent to an additional vertex. Together these claims are contradictory. Therefore if G is not one of the exceptional graphs in Figure 1 or Figure 2 and v is a vertex of G with $d(v) = 3$, then v must lie on a K_4. We have proved Claim 2.4/1.3.5. ∎

By Theorem 2.1 and Claim 2.4/1.1, $d(v) < 6$. Hence since G is 3-connected, we know that $3 \leq d(v) \leq 5$ for all v in $V(G)$. By Claim 2.4/1.3.3 we know that every vertex of degree five must lie on a K_4. By Claim 2.4/1.3.4 we know that every vertex of degree four must lie on a K_4. By Claim 2.4/1.3.5 we know that every vertex of degree three must lie on a K_4. Therefore we have shown that if G is not one of the exceptional graphs in Figure 1 or Figure 2, then every vertex of G must lie on a K_4, and so we have proved Claim 2.4/1.3. ∎

Claim 2.4/1.4: If G is not one of the exceptional graphs in Figure 1 or Figure 2, then any two K_4's in G must be disjoint.

Proof of Claim 2.4/1.4: Let G be a graph that fulfills the hypothesis of the claim. Suppose, by way of contradiction, that there exist two K_4's, $K_4(1)$ and $K_4(2)$, in G that are not disjoint. By planarity, these two K_4's may share at most three vertices. Recall that by Theorem 2.1, $d(v) \leq 6$ for all v in $V(G)$. Suppose $K_4(1)$ and $K_4(2)$ share exactly one vertex, v. Then v must be adjacent to three vertices of $K_4(1)$ and three vertices of $K_4(2)$ that are all distinct, since $K_4(1)$ and $K_4(2)$ share exactly one vertex. But then $d(v) = 6$, and so by Claim 2.4/1.1, G must be the exceptional graph in Figure 1(a) or Figure 1(b). This contradicts the fact that G is not a graph in Figure 1, and so $K_4(1)$ and $K_4(2)$ cannot share exactly one vertex.

Suppose $K_4(1)$ and $K_4(2)$ share exactly two vertices, v and u. Since v and u are in a K_4 together, we know that $v \sim u$. Let w and x be the remaining two vertices of $K_4(1)$, such that w is interior to the uvx-face. Let y and z be the remaining two vertices of $K_4(2)$, such that y is interior to the uvz-face. See Figure 110(a) for an illustration. Now $d(u) = d(v) = 5$ and so by Claim 2.4/1.2, neither w nor y may grow. Since G is 3-connected, $\{x, z\}$ is not a 2-cut. Therefore G cannot contain any

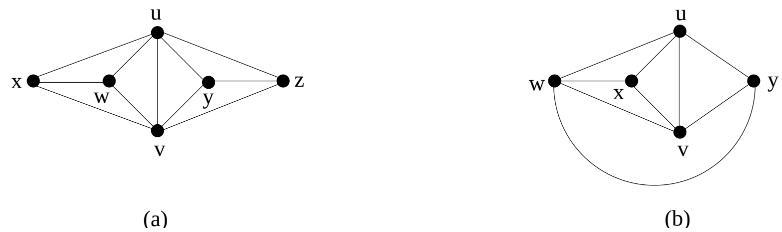


Figure 110: Proving that any two K_4's of G are disjoint.

additional vertices, and we must have $x \sim z$. But then G is not well-covered, since $\{u\}$ and $\{w, y\}$ are both maximal independent sets of G. Hence $K_4(1)$ and $K_4(2)$ cannot share exactly two vertices.

Suppose $K_4(1)$ and $K_4(2)$ share exactly three vertices, v, u and w. Let x be the fourth vertex of $K_4(1)$ and y be the fourth vertex of $K_4(2)$. See Figure 110(b) for an illustration. Note that the graph is not well-covered, since $\{u\}$ and $\{x, y\}$ are both maximal independent, and so the graph must grow. Since G is 3-connected, $\{w, y\}$ is not a 2-cut, and so either u or v must be adjacent to an additional vertex. Without loss of generality, suppose that u is adjacent to an additional vertex; call it z. To prevent $\{ux, uy, uz\}$ from forming a claw at u, we must have $y \sim z$. Now $d(u) = 5$, and so by Claim 2.4/1.2, x cannot grow and $d(x) = 3$. Since G is 3-connected, $\{u, y\}$ is not a 2-cut and so there must be a path from z to w that does not pass through either u or y. Thus either w is adjacent to z, or w is adjacent to an additional vertex in the exterior face.

Suppose $w \sim z$. Then $d(w) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, w cannot grow. Thus there are no additional vertices in the vwy-face; otherwise $\{v, y\}$ would be a 2-cut, contradicting the fact that G is 3-connected. Furthermore, there are no additional vertices in the wyz-face; otherwise $\{x, z\}$ would be a 2-cut, contradicting the fact that G is 3-connected. Hence the graph cannot grow. But it is not well-covered, since $\{u\}$ and $\{x, z\}$ are both maximal independent sets of the graph. Therefore $w \nsim z$.

Suppose w is adjacent to an additional vertex in the exterior face; call it t. To prevent $\{wx, wy, wt\}$ from forming a claw at w, we must have $t \sim y$. To prevent $\{yv, yz, yt\}$ from forming a claw at y, we must have $t \sim z$. Note that $d(w) = d(y) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, w and y cannot grow. By Claim 2.4/1.3, z must lie on a K_4. Since two of z's neighbors (u and y) cannot grow, z must then be adjacent to an additional vertex; call it s. To prevent $\{zu, zt, zs\}$ from forming a claw at z, we must have $t \sim s$. But then $\{z, t\}$ is a 2-cut, separating s from the rest of the graph and contradicting the fact that G is 3-connected. Therefore w is not adjacent to an additional vertex.

Hence $K_4(1)$ and $K_4(2)$ cannot share exactly three vertices.

Therefore $K_4(1)$ and $K_4(2)$ share none of their vertices, and any two K_4's in G must be disjoint. ■

Claim 2.4/1.5: If an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$.

Proof of Claim 2.4/1.5: Let v be an exterior vertex of a K_4 in G that is adjacent to two vertices u and w on two other K_4's. Suppose, by way of contradiction, that $u \nsim w$. Let x be the interior vertex of v's K_4. Then $\{vx, vu, vw\}$ is a claw at v, contradicting the fact that G is claw-free. Thus if an exterior vertex of a K_4 is joined to two vertices u and w on two other K_4's, then $u \sim w$. ■

If G is planar, 3-connected, claw-free and well-covered, by Theorem 2.1 $d(v) \leq 6$ for all v in $V(G)$. By Claim 2.4/1.1, if G has a vertex of degree six, G is one of two graphs in Figure 1. By Claim 2.4/1.3, if G is not a graph in Figure 1 or Figure 2, then every vertex of G must lie on a K_4. By Claim 2.4/1.4, these K_4's must be distinct. By Claim 2.4/1.5, if an exterior vertex of a K_4 is joined to two vertices u

and w on two other K_4's, then $u \sim w$. Thus if G is planar, 3-connected, claw-free and well-covered, then G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$. Hence we have proved Claim 2.4/1. ■

Claim 2.4/2: If G is one of the exceptional graphs in Figure 1 or Figure 2 or G is in the class $\mathcal{G}$, then G is planar, 3-connected, claw-free and well-covered.

Proof of Claim 2.4/2: We leave it to the reader to check that the graphs in Figures 1 and 2 are planar, 3-connected, claw-free and well-covered. Clearly K_4 is planar, 3-connected, claw-free and well-covered. So suppose that G is a graph the class $\mathcal{G}$ that is not K_4. By definition of $\mathcal{G}$, G is planar and 3-connected.

Suppose, by way of contradiction, that G contains a claw at v, a vertex of G, with vertices u, w and x. Then each of u, w and x are adjacent to v, but $\{u, w, x\}$ is an independent set. By Theorem 2.1 and Claim 2.4/1.1, we know that $d(v) \leq 5$. Thus v has at most two neighbors outside its K_4. Since any two vertices within the same K_4 must be adjacent and there exists a claw at v, exactly two of $\{u, w, x\}$ must be neighbors of v from outside v's K_4, and these neighbors must be on distinct K_4's. Suppose u is the vertex in v's K_4, and w and x are neighbors in other K_4's. But then by definition of $\mathcal{G}$, if v is joined to two vertices w and x on two other K_4's, then $w \sim x$. Thus there is no claw at v. Hence G is claw-free.

Finally, we must show that G is well-covered. Clearly, any maximal independent set of G may contain at most one vertex from each K_4. By planarity and 3-connectivity, there is one vertex from each K_4 that is not connected to any other K_4's. Thus every maximal independent set of G must contain exactly one vertex from each K_4. Therefore every maximal independent set has the same cardinality, and therefore G is well-covered.

Thus if G is one of the exceptional graphs in Figure 1 or Figure 2 or G is in the class $\mathcal{G}$, then G is planar, 3-connected, claw-free and well-covered. ■

Let G be a planar, 3-connected graph. Then by Claim 2.4/1 and Claim 2.4/2, G is claw-free and well-covered if and only if G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$. Therefore we have proved Theorem 2.4. ■

Corollary 2.5: Let G be a planar, 3-connected graph. Then G is claw-free and well-dominated if and only if G is one of the exceptional graphs in Figure 1 or Figure 2(a)-(j), or G is in the class $\mathcal{G}$.

Proof: Let G be a planar, 3-connected graph. By Theorem 2.4, G is claw-free and well-covered if and only if G is one of the exceptional graphs in Figure 1 or Figure 2, or G is in the class $\mathcal{G}$. By Lemma 1.1, the set of claw-free, well-dominated graphs is a subset of the claw-free, well-covered graphs. Thus we must only determine which of the exceptional graphs in Figure 1, Figure 2, and the class $\mathcal{G}$ are also well-dominated.

We leave it to the reader to check that the all of the exceptional graphs in Figure 1, and the exceptional graphs in (a)-(j) of Figure 2 are well-dominated. Note that the graph in Figure 2(k) is not well-dominated. This graph is a K_4 and a K_3 with the exterior vertices of the K_4 joined to the K_3 by a matching. We may minimally dominate this graph with two vertices by choosing one vertex from the K_4 and one from the K_3. We may also minimally dominate this graph by choosing all three of the exterior vertices of the K_4. Thus it is not well-dominated. Also note that the graph in Figure 2(l) is not well-dominated. This graph is formed by two K_3's joined by a matching. We may minimally dominate this graph with two vertices by choosing one vertex from each of the K_3's. We may also minimally dominate this graph by choosing all three of the vertices from one of the K_3's. Thus it is not well-dominated.

Finally we must show that all of the graphs in the class $\mathcal{G}$ are well-dominated.

Suppose G is in the class $\mathcal{G}$. Clearly if a set of vertices of G contains one vertex from each K_4, then it is dominating. By planarity and 3-connectivity, there is one vertex from each K_4 that is not connected to any other K_4's. Thus every minimal dominating set of G must contain exactly one vertex from each K_4. Therefore every minimal dominating set has the same cardinality, and therefore G is well-dominated.

Therefore, G is claw-free and well-dominated if and only if G is one of the exceptional graphs in Figure 1 or Figure 2(a)-(j), or G is in the class $\mathcal{G}$. ■

In order to ensure that the graphs in the class $\mathcal{G}$ are 3-connected, they must have the properties described in the following lemmas.

Lemma 2.6: Let G be a graph in the class $\mathcal{G}$ containing at least two K_4's. Then each of the three exterior vertices of each K_4 of G must be adjacent to one vertex of at least one other K_4.

Proof: Suppose $\{u, v, w\}$ are the exterior vertices of a K_4 of G, and x is the interior vertex. By way of contradiction, suppose that v is not adjacent to any vertices other than the set $\{u, w, x\}$. Then $\{u, w\}$ is a 2-cut, separating v and x from the rest of the graph and contradicting the fact that G is 3-connected. Hence v must be adjacent to a vertex of at least one other K_4. ■

Lemma 2.7: Let G be a graph in the class $\mathcal{G}$ containing at least three K_4's. Then no two vertices from the same K_4 may be adjacent to the same vertex in another K_4.

Proof: Suppose G fulfills the hypotheses of the lemma. Suppose $\{u, v, w\}$ are the exterior vertices of a K_4 of G, and x is the interior vertex. By way of contradiction, suppose that u and w are both adjacent to y, a vertex in another K_4. Suppose $\{y, z, t\}$ are the exterior vertices of y's K_4, and s is the interior vertex. Then $d(y) = 5$ and by Theorem 2.1 and Claim 2.4/1.1, y cannot grow. By definition of $\mathcal{G}$, s cannot grow

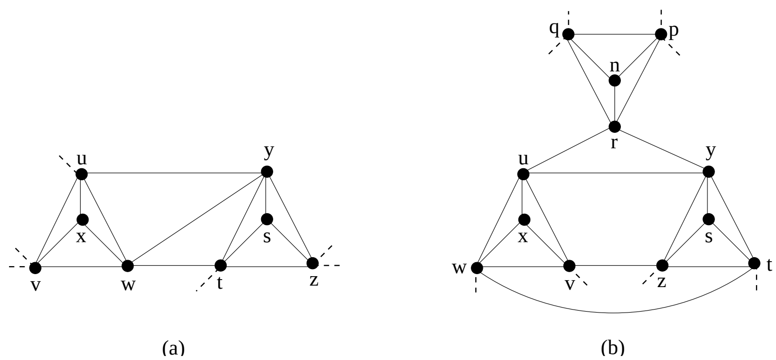


Figure 111: Proving properties of graphs in the class $\mathcal{G}$.

and $d(s) = 3$. Since G is 3-connected, $\{v, y\}$ is not a 2-cut, and so u or w must grow. Without loss of generality, suppose that w grows. The vertex w cannot be adjacent to an additional vertex, or this additional vertex together with x and y will form a claw at w, contradicting the fact that G is claw-free. Thus w must be adjacent to either t or z. Without loss of generality, suppose that $w \sim t$. Now $d(w) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, w cannot grow. By definition of $\mathcal{G}$, x cannot grow. See Figure 111(a) for an illustration. Since G is 3-connected, $\{v, z\}$ is not a 2-cut, and so either u or t must grow. Without loss of generality, suppose u grows. The vertex u cannot be adjacent to an additional vertex, or this additional vertex, together with x and y will form a claw at w, contradicting the fact that G is claw-free. Thus either u is adjacent to t or u is adjacent to z.

Suppose $u \sim t$. Then $d(u) = d(t) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, neither u nor t may grow. But then $\{y, t\}$ is a 2-cut, separating z and s from the rest of the graph and contradicting the fact that G is well-covered.

Suppose $u \sim z$. Then $d(u) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, u cannot grow. Now t cannot be adjacent to an additional vertex, since this additional vertex, together with w and s would form a claw at t, contradicting the fact that G is claw-free. Also z cannot be adjacent to an additional vertex; otherwise this additional neighbor, together with u and s would form a claw at z, contradicting the fact that G is claw-free. Thus since G contains at least three K_4's and is connected, v must be adjacent to an additional vertex. But then v is a cut-vertex, contradicting the fact that G is 3-connected. Hence $u \nsim z$.

Therefore, no two vertices from the same K_4 may be adjacent to the same vertex in another K_4. ∎

Lemma 2.8: Let G be a graph in the class $\mathcal{G}$ containing at least three K_4's. Then each K_4 is joined by no more than two edges to any other K_4.

Proof: Suppose that G fulfills the hypotheses of the lemma. By Theorem 2.1 and Claim 2.4/1.1, the degree of any vertex of G is at most five, and so each vertex is adjacent to at most two vertices in another K_4. By Lemma 2.7, no two vertices of the same K_4 may share a neighbor in another K_4. Thus every K_4 may have at most three edges joining it to another K_4, and the edges must form a matching.

Suppose $\{u, v, w\}$ are the exterior vertices of one K_4 of G, and x is the interior vertex of that K_4. Suppose $\{y, z, t\}$ are the exterior vertices of another K_4, and s is the interior vertex of the second K_4. By way of contradiction, suppose that there exist three edges between these two K_4's. Without loss of generality, suppose $u \sim y$, $v \sim z$ and $w \sim t$. By claw-freedom, if either vertex in the set $\{u, y\}$ is adjacent to an additional vertex, the other vertex in that set must be adjacent to the additional vertex as well, and similarly for the sets $\{v, z\}$ and $\{w, t\}$. Recall by hypothesis, there exists at least one more K_4 in G. If neither vertex set $\{u, y\}$ or $\{v, z\}$ grows, then $\{w, t\}$ is a 2-cut, contradicting the fact that G is 3-connected. Thus without loss of generality, suppose that u and y share an additional neighbor; call it r, where r is a vertex of a third K_4. Suppose $\{r, q, p\}$ are the exterior vertices of r's K_4, and n is the interior vertex of r's K_4. See Figure 111(b) for an illustration. Now $d(u) = d(y) = d(r) = 5$, and so by Theorem 2.1 and Claim 2.4/1.1, u, y and r cannot grow. Since G is 3-connected, r is not a cut-vertex and so there must exist a path from w and t to q and p that does not pass through r. Either w and t are both adjacent to q or p, or w and t share an additional neighbor. The argument for these cases is the same, so suppose w and t share a neighbor, where this neighbor is either q, p, or an additional vertex. Then the degree of each of w, t and this neighbor is five, and so by Theorem 2.1 and Claim 2.4/1.1, w, t and this neighbor cannot grow. But then r together with this neighbor is a 2-cut, separating either n and p (if the neighbor is q), n and q (if the neighbor is p), or $\{u, v, w, x, y, z, t, s\}$ (if the neighbor is an additional vertex) from the rest of the graph, and contradicting the fact that G is 3-connected. Hence neither w nor t is adjacent to either q or p, and w and t do not share an additional neighbor.

Therefore at most two edges join any two K_4's of G. ■

Lemma 2.9: Let G be a graph in the class $\mathcal{G}$ containing at least two K_4's (i.e. G is not K_4). Then either G is one of the graphs in Figure 112, or each K_4 in G is joined to at least three other K_4's.

Proof: Suppose that G fulfills the hypotheses of the lemma.

Claim 2.9.1: If G is a graph in the class $\mathcal{G}$ containing at exactly two K_4's, then G is one of the graphs in Figure 112(a)-(d).

Proof of Claim 2.9.1: Suppose that G contains exactly two disjoint K_4's. Let $\{u, v, w\}$ be the exterior vertices of one K_4 of G, and x be the interior vertex of that K_4. Let $\{y, z, t\}$ be the exterior vertices of the other K_4, and s be the interior vertex of the second K_4. By Lemma 2.6, each vertex of $\{u, v, w\}$ must be adjacent to a vertex of $\{y, z, t\}$ and vice versa. Thus there must exist at least three edges between $\{u, v, w\}$ and $\{y, z, t\}$. By Theorem 2.1 and Claim 2.4/1.1, the degree of any vertex of G is at most five. Thus each of $\{u, v, w\}$ may be adjacent to at most

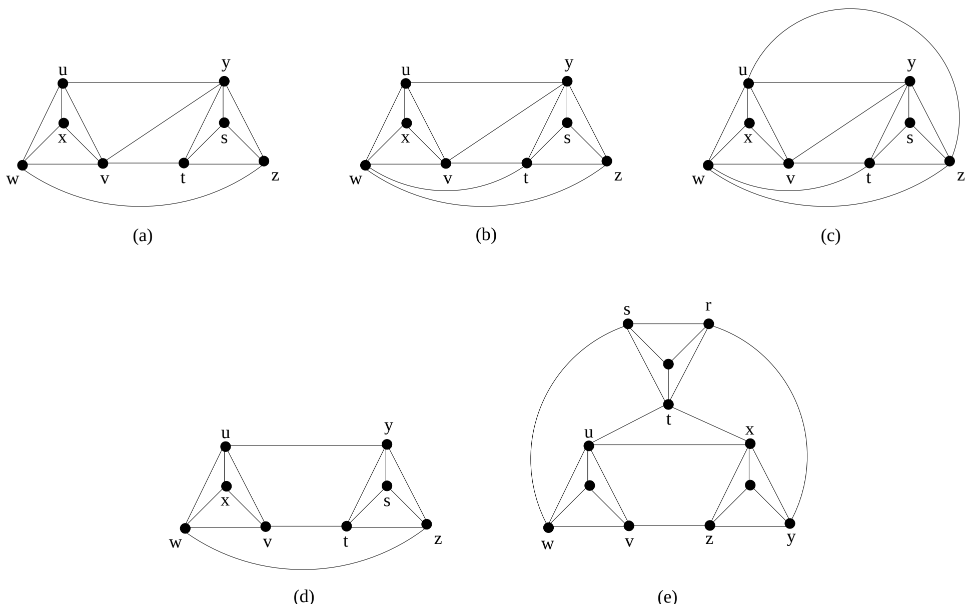


Figure 112: Small graphs in the class $\mathcal{G}$.

two of $\{y, z, t\}$ and vice versa, which implies that there may be at most six edges between the two vertex sets. Without loss of generality, suppose that u is adjacent to y. Either v is also adjacent to y, or v is adjacent to z or t (i.e. $v \nsim y$).

Suppose that $v \sim y$. Now $d(y) = 5$ and so y cannot be adjacent to any other vertices by Theorem 2.1 or Claim 2.4/1.1. Thus w must be adjacent to either t or z by Lemma 2.6. Suppose, without loss of generality, that $w \sim z$. Since G is 3-connected, $\{y, z\}$ is not a 2-cut, and so there must be a path from t to v and w that does not pass through either y or z. Thus either $t \sim v$ or $t \sim w$ (since there are only two K_4's in G). Suppose $t \sim v$. Then we have the graph shown in Figure 112(a), which is a member of the class $\mathcal{G}$. Suppose $t \sim w$. Since G is 3-connected, $\{w, y\}$ is not a 2-cut, and so there must be a path from v to t that does not pass through either w or y. Note that $v \nsim z$ by planarity. Thus we must have $t \sim v$. Then we have the graph shown in Figure 112(b), which is a member of the class $\mathcal{G}$. Can this graph grow? Note that $d(y) = d(v) = d(w) = d(t) = 5$, and so none of these vertices can grow by Theorem 2.1 and Claim 2.4/1.1. Thus the only addition we may have is $u \sim z$. Then we have the graph shown in Figure 112(c), which is a member of the class $\mathcal{G}$. This graph can grow no further.

Suppose $v \nsim y$. By symmetry, we may assume from this that v and u do not share any neighbors in the second K_4, and thus we may generalize by symmetry to say that no two vertices from one of the K_4's may share a neighbor in the other K_4. Then by Lemma 2.6, we must have a matching between the vertex set $\{u, v, w\}$ and the vertex set $\{y, z, t\}$, and this graph cannot grow. Suppose without loss of generality, we have $u \sim y$, $v \sim t$ and $w \sim z$. Then we have the graph shown in Figure 112(d), which is a member of the class $\mathcal{G}$.

Therefore if G is a graph in the class $\mathcal{G}$ containing at exactly two K_4's, then G is one of the graphs in Figure 112(a)-(d). ∎

Claim 2.9.2: If G is a graph in the class $\mathcal{G}$ containing at exactly three K_4's, then G is the graph shown in Figure 112(e).

Proof of Claim 2.9.2: Suppose that G contains exactly three disjoint K_4's. Let $\{u, v, w\}$ be the exterior vertices of the first K_4; call it $K_4(1)$. Let $\{x, y, z\}$ be the exterior vertices of the second K_4; call it $K_4(2)$. Let $\{t, s, r\}$ be the exterior vertices of the third K_4; call it $K_4(3)$. By Lemma 2.6, every exterior vertex of a K_4 in G must be adjacent to an exterior vertex of another K_4. By Lemma 2.7, no two vertices of the same K_4 can be adjacent to the same vertex of another K_4. Furthermore, by Lemma 2.8, there may be at most two edges between any two K_4's. Thus since G contains exactly three K_4's, two exterior vertices of $K_4(1)$ must have edges to distinct exterior vertices of either $K_4(2)$ or $K_4(3)$, say $K_4(2)$ without loss of generality, and the third exterior vertex must have an edge to the other K_4, $K_4(3)$. A similar situation must exist for the exterior vertices of $K_4(2)$ and $K_4(3)$. Without loss of generality, say $u \sim x$, $v \sim z$ and $w \sim s$. By Lemma 2.6, y must be adjacent to an exterior vertex in either $K_4(1)$ or $K_4(2)$. Since there are already two edges between $K_4(1)$ and $K_4(2)$, y must be adjacent to a vertex in $K_4(3)$ by Lemma 2.8. Note that $y \nsim w$ by Lemma 2.8. Therefore $y \nsim s$; otherwise the interior vertex of $K_4(3)$ together with w and y would form a claw at s, contradicting the fact that G is claw-free. Thus y is adjacent to either t or r. Without loss of generality, say $y \sim r$. By Lemma 2.6, t must be adjacent to an exterior vertex of either $K_4(1)$ or $K_4(2)$. By Lemma 2.7, $t \nsim w$ and $t \nsim y$. Without loss of generality, suppose $t \sim u$. To prevent $\{uv, ut, ux\}$ from forming a claw at u, we must have $t \sim x$, since $t \nsim v$ by Lemma 2.7. Then we have the graph shown in Figure 112(e), which is a member of the class $\mathcal{G}$. Note that there are exactly two edges between each of the K_4's, and therefore the graph cannot grow. Therefore if G is a graph in the class $\mathcal{G}$ containing at exactly three K_4's, then G is the graph shown in Figure 112(e). ∎

Claim 2.9.3: If G is a graph in the class $\mathcal{G}$ containing greater than three K_4's, then each K_4 in G is joined to at least three other K_4's.

Proof of Claim 2.9.3: Suppose that G contains at least four disjoint K_4's. Let $\{u, v, w\}$ be the exterior vertices of the first K_4; call it $K_4(1)$. Suppose by way of contradiction, that $K_4(1)$ is joined to less than three other K_4's. By Lemma 2.6, each of u, v and w must be adjacent to a vertex in another K_4. By Lemma 2.8, there may exist at most two edges between any two K_4's. Thus $K_4(1)$ must be joined to exactly two K_4's; call them $K_4(2)$ and $K_4(3)$. Let $\{x, y, t\}$ be the exterior vertices of $K_4(2)$, and let $\{z, s, r\}$ be the exterior vertices of $K_4(3)$. Without loss of generality, suppose u and v are adjacent to exterior vertices of $K_4(2)$ (by Lemma 2.7 they must be distinct), and w is adjacent to an exterior vertex of $K_4(3)$. So assume $u \sim x$, $v \sim y$, and $w \sim z$. Note that by Lemma 2.8, there are no more edges joining $K_4(1)$ and $K_4(2)$. Since G is 3-connected, $\{w, t\}$ is not a 2-cut, and so there must be a path from $\{u, v\}$ to $\{s, r\}$ that does not pass through either w or t. Note that u and v

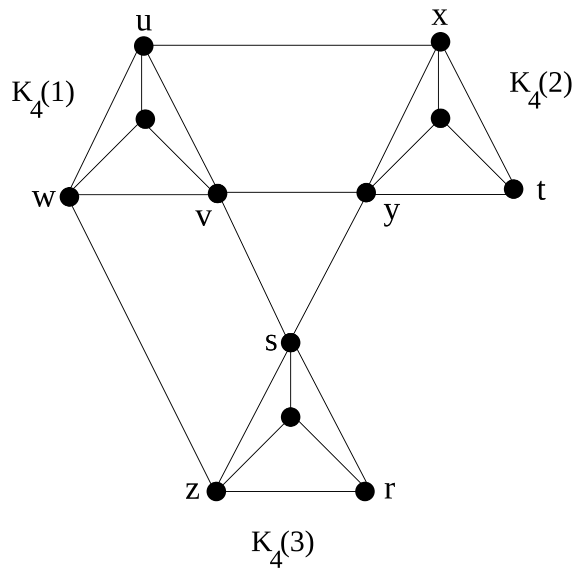


Figure 113: Proving properties of graphs in the class $\mathcal{G}$.

are symmetric and so we may assume, without loss of generality, that the path from $\{u, v\}$ to $\{s, r\}$ leaves $K_4(1)$ at v. Since $K_4(1)$ is joined to only two other K_4's, v is not adjacent to an additional vertex. Note that by claw-freedom, if v is adjacent to s or r, then y must be adjacent to that vertex as well, and vice versa. Thus either v and y are both adjacent to s, or v and y are both adjacent to y. Without loss of generality, suppose that both v and y are adjacent to s. See Figure 113 for an illustration. By Lemma 2.8, there are no additional edges between $K_4(1)$ and $K_4(3)$, and so u and w cannot grow. Thus by claw-freedom, x and z cannot grow. But then $\{r, t\}$ is a 2-cut, separating the vertices of $K_4(1)$ and the other vertices of $K_4(2)$ and $K_4(3)$ from the rest of the graph, and contradicting the fact that G is claw-free. Hence if G is a graph in the class $\mathcal{G}$ containing at greater than three K_4's, then each K_4 in G is joined to at least three other K_4's. ■

Therefore by Claims 2.9.1, 2.9.2, and 2.9.3, if G is a graph in the class $\mathcal{G}$ containing at least two K_4's (i.e. G is not K_4), then either G is one of the graphs in Figure 112, or each K_4 in G is joined to at least three other K_4's. ■

There are an infinite number of planar, 3-connected, claw-free, well-covered (and well-dominated) graphs. To show that this is true, I will discuss several techniques for enlarging a graph of $\mathcal{G}$ with n K_4's to a graph of $\mathcal{G}$ with $n+3$ or $n+1$ K_4's. To prove that the resulting graphs are 3-connected, we will need the following terminology and theorem, which is an extension of Menger's Theorem.

Definition [20]: Given a vertex x and a set U of vertices, an **x,U-fan** is a set of paths from x to U such that any two of them share only the vertex x.

Theorem 2.10 [4]: A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x and U with $|U| \geq k$, it has an x, U-fan of size k.

Technique 1: Let G be a graph in $\mathcal{G}$ on n K_4's containing a vertex of degree five, call it u. Then u must be a vertex of a K_4 that is adjacent to two other vertices on two other distinct K_4's, by Lemma 2.7. Call these neighbors of u in other K_4's v and w. By definition of $\mathcal{G}$, $v \sim w$. Delete the edges $\{uv, uw, vw\}$ from G and insert three new K_4's. Join each of u, v and w to two distinct exterior vertices on two distinct new K_4's so that each of u, v and w has edges to a unique pair of new K_4's, and add a set of edges necessary for claw-freedom. Each new K_4 then has one remaining exterior vertex of degree three; form a triangle with these vertices so that they all have degree five. Call the resulting graph G^+. See Figure 114 for an illustration. By Theorem 2.10, there exist three vertex disjoint paths from any vertex in $V(G)-\{u, v, w\}$ to the set $U = \{u, v, w\}$. Thus if x is a vertex of $V(G)-\{u, v, w\}$ and y is a new vertex, there exist three vertex disjoint paths from x to y, since the paths from x to U may be extended in a vertex disjoint way to y. We leave it to the reader to check that there are three vertex disjoint paths in G^+ between any two vertices of U, between a vertex of U and a new vertex, and between any two new vertices. Thus G^+ is 3-connected. It is straightforward to check that G^+ is claw-free, planar and well-covered, and so G^+ is still a graph in $\mathcal{G}$, and it contains twelve more

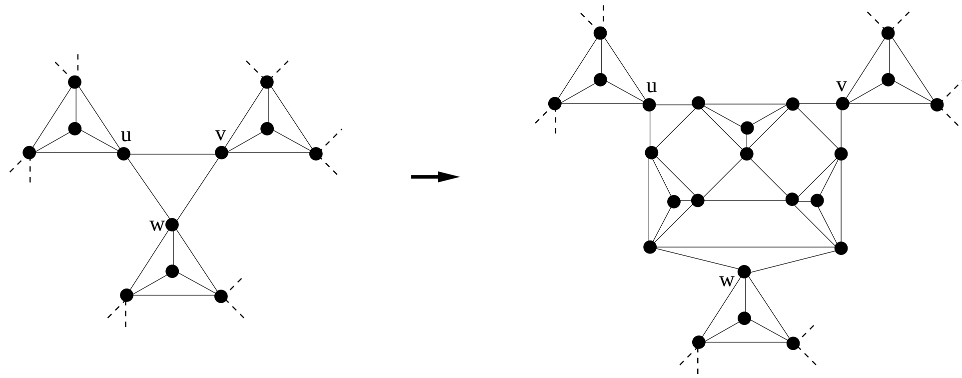


Figure 114: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 1.

vertices than G does.

Technique 2: Let G be a graph in $\mathcal{G}$ on n K_4's containing a K_4 with two vertices of degree four; call them u and v. By Lemma 2.7, the fourth neighbors of u and v (i.e. the neighbors of u and v that are in another K_4) are distinct; call them w and x, such that $u \sim w$ and $v \sim x$. Note that w and x cannot be in the same K_4; otherwise the third exterior vertex of the K_4 containing u and v together with the third exterior vertex of the K_4 containing w and x would form a 2-cut, separating u, v, w, x and the two interior vertices of the two K_4's from the rest of the graph. Delete the edge vx from G and insert a new K_4. Let the exterior vertices of the new K_4 be y, z and t, and join the new K_4 with the following edges: uy, wy, zv, tx. Call the resulting graph G^+. See Figure 115 for an illustration. By Theorem 2.10, there exist three vertex disjoint paths from any vertex in $V(G) - \{u, v, x\}$ to the set $U = \{u, v, x\}$. Thus if s is a vertex of $V(G) - \{u, v, x\}$ and r is a new vertex, there exist three vertex disjoint paths from s to r, since the paths from s to U may be extended in a vertex disjoint way to r. We leave it to the reader to check that there are three vertex disjoint paths in G^+ between any two vertices of U, between a vertex of U and a new vertex, and between any two new vertices. Thus G^+ is 3-connected. It is straightforward to check that G^+ is claw-free, planar and well-covered, and so G^+ is

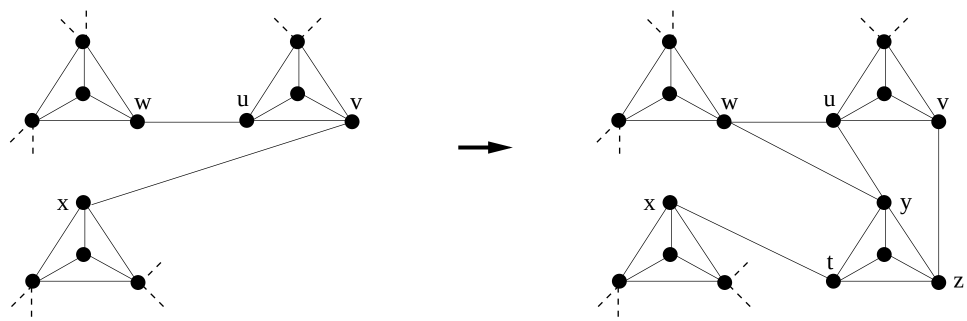


Figure 115: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 2.

still a graph in $\mathcal{G}$, and it contains four more vertices than G does.

Technique 3: Let G be a graph in $\mathcal{G}$ on n K_4's containing at least three edges with the property that the vertices of the edge have degree four and the edge joins two K_4's (i.e. the edge is not contained within a K_4). Insert a new K_4, and join it to G so that each of the exterior neighbors of the new K_4 is joined to the two vertices of one of the edges of G with the property. See Figure 116 for an illustration of building a graph in $\mathcal{G}$ with four K_4's from the graph in Figure 112(e) using Technique 3. The three edges we use to join the new K_4 are shown in bold in the original graph. Again we use Theorem 2.10 to show that the resulting graph is 3-connected. Here let U be a set of three vertices containing one vertex from each of the three edges with the property used to build the new graph. It is straightforward to check that the new graph is claw-free, planar and well-covered, and so this graph is still a graph in $\mathcal{G}$, and it contains four more vertices than G does.

Note that by Lemma 2.6, each graph G in $\mathcal{G}$ either contains a vertex of degree five, or all the exterior vertices of the K_4's in G have degree four. Thus at least one of these three techniques can be applied to any given graph in $\mathcal{G}$ to build a larger graph (i.e. a graph with more vertices) in $\mathcal{G}$. Hence the class of graphs $\mathcal{G}$ contains an infinite number of graphs. Therefore there are an infinite number of

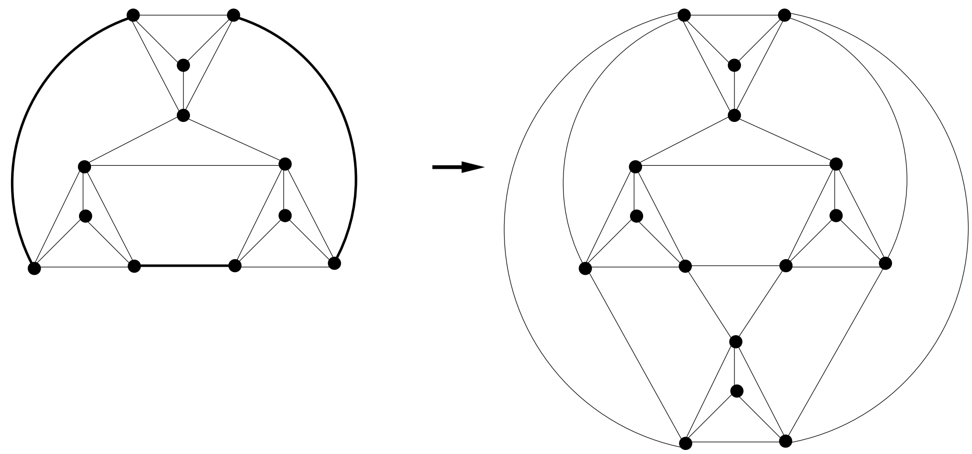


Figure 116: Building a graph in the class $\mathcal{G}$ from a smaller graph in $\mathcal{G}$ using Technique 3.

planar, 3-connected, claw-free, well-covered (and well-dominated) graphs.

BIBLIOGRAPHY

1. S. R. Campbell, M. N. Ellingham and G. F. Royle. A characterisation of well-covered cubic graphs. *J. Combin. Math. Combin. Comput.*, **13** (1993) 193-212.

2. S. R. Campbell and M. D. Plummer. On well-covered 3-polytopes. *Ars Combin.*, **25A** (1988) 215-242.

3. V. Chvátal and P. J. Slater. A note on well-covered graphs. *Quo Vadis, Graph Theory? Ann. Discrete Math.*, **55** (1993) 179-182.

4. G. A. Dirac. In abstrakten Graphen vorhande vollständigene 4-Graphen und ihre Unterteilungen. *Math. Nachr.*, **22** (1960) 61-85.

5. A. Finbow, B. Hartnell, and R. J. Nowakowski. Well-dominated graphs: a collection of well-covered ones. *Ars Combin.*, **25A** (1988) 5-10.

6. J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts. On graphs having domination number half their order. *Period. Math. Hungar.*, **16** (1985) 287-293.

7. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs.* New York: Marcel Dekker, Inc. (1998).

8. T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, ed. *Domination in Graphs: Advanced Topics.* New York: Marcel Dekker, Inc. (1998).

9. B. L. Hartnell and M. D. Plummer. On 4-connected claw-free well-covered graphs. *Discrete Appl. Math.*, **64** (1996) 57-65.

10. E. L. C. King. *Some open problems in domination theory.* Qualifying Paper, Department of Mathematics, Vanderbilt University (1999).

11. C. Payan and N.H. Xuong. Domination-balanced graphs. *J. Graph Theory*, **6** (1982) 23-32.

12. M. R. Pinter. A construction which yields strongly well-covered graphs. *Congr. Numer.*, (Proceedings of the Twenty-eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing), **128** (1997) 33-43.

13. M. R. Pinter. Planar strongly well-covered graphs. *Graph Theory, Combinatorics, and Algorithms, Vol. 1, 2.* Wiley-Intersci. Publ. (1995) 869-879.

14. M. R. Pinter. Strongly well-covered graphs. *Discrete Math.*, **132** (1994) 231-246.

15. M. R. Pinter. *W_2 graphs and strongly well-covered graphs: two well-covered graph subclasses.* Ph.D. Dissertation, Department of Mathematics, Vanderbilt University (1991).

16. M. D. Plummer. Claw-free maximal planar graphs. *Congr. Numer.*, **72** (1990) 9-23.

17. B. Randerath and P. D. Vestergaard. A note on well-covered graphs with odd girth. *Aalborg University Report R-99-2019*, (November 1999) 1-8.

18. R. S. Sankaranarayana and L. K. Stewart. Complexity results for well-covered graphs. *Networks*, **22** (1992) 247-262.

19. J. Topp and L. Volkmann. On domination and independence numbers of graphs. *Resultate Math.*, **17** (1990) 333-341.

20. D. B. West. *Introduction to Graph Theory, second edition.* Prentice-Hall, Inc. (2001).