Prove the following.
Theorem
Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.
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Why is that?

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## Theorem

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Why is that?

$$
C=\bar{B}=\overline{\bar{A}}
$$

(replace $B$ with $\bar{A}$ in the last equality) Proving $A=C$ is the same as showing $A=\overline{\bar{A}}$.

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.

## Definition (Set equality)

Let $A$ and $B$ be sets. Then $A=B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

$$
\left[\begin{array}{cccc}
\text { Given } A=B, \text { we know } \\
\circ & x \in A \Rightarrow x \in B & \circ \\
\circ & x \in B \Rightarrow x \in A & \circ \\
\circ & x \notin A \Rightarrow x \notin B & 0 \\
\circ & x \notin B \Rightarrow x \notin A & \circ
\end{array}\right.
$$

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$$
\left|\begin{array}{cccc}
\text { Given } B=\bar{A} \text {, we know } \\
\circ & x \in B \Rightarrow x \in \bar{A} & \circ \\
\circ & x \in \bar{A} \Rightarrow x \in B & \circ \\
\circ & x \notin B \Rightarrow x \notin \bar{A} & \circ \\
\circ & x \notin \bar{A} \Rightarrow & x \notin B & \circ
\end{array}\right|
$$

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Let $A$ and $B$ be sets. Then $A=B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

$$
\left\lvert\, \begin{array}{ccc}
\text { Given } C=\bar{B} \text {, we know } \\
\circ & x \in C \Rightarrow x \in \bar{B} & \circ \\
\circ & x \in \bar{B} \Rightarrow x \in C & \circ \\
\circ & x \notin C \Rightarrow x \notin \bar{B} & 0 \\
\circ & x \notin \bar{B} \Rightarrow x \notin C & \circ
\end{array}\right.
$$

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.
Definition (Set complement)
Let $A$ be a set. Then $\bar{A}=\{x: x \notin A\}$.

$$
\left[\begin{array}{cccc}
\text { Given a set } A \text {, we know } \\
0 & x \in A \Rightarrow x \notin \bar{A} & \circ \\
0 & x \notin \bar{A} \Rightarrow & x \in A & \circ \\
0 & x \notin A \Rightarrow x \in \bar{A} & 0 \\
0 & x \in \bar{A} \Rightarrow & x \notin A & 0
\end{array}\right]
$$

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.

## Definition (Set complement)

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$$
\left(\left.\begin{array}{llll}
\text { Given the set } A \text {, we know } \\
\circ & x \in A \Rightarrow & x \notin \bar{A} & \circ \\
\circ & x \notin \bar{A} \Rightarrow & x \in A & \circ \\
\circ & x \notin A \Rightarrow & x \in \bar{A} & \circ \\
\circ & x \in \bar{A} \Rightarrow & x \notin A & \circ
\end{array} \right\rvert\,\right.
$$

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.
Definition (Set complement)
Let $A$ be a set. Then $\bar{A}=\{x: x \notin A\}$.

$$
\left|\begin{array}{lll}
\text { Given the set } B \text {, we know } \\
0 & x \in B \Rightarrow x \notin \bar{B} & \circ \\
\circ & x \notin \bar{B} \Rightarrow x \in B & \circ \\
0 & x \notin B \Rightarrow x \in \bar{B} & \circ \\
\circ & x \in \bar{B} \Rightarrow x \notin B & \circ
\end{array}\right|
$$

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.

## Proof.

We use a double inclusion argument to prove that $A=C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

Theorem (Double inclusion principle)
Let $A$ and $B$ be sets. Then $A=B$ if and only if $A \subseteq B$ and $B \subseteq A$.

## Theorem

Let $A, B$, and $C$ be sets where $B=\bar{A}$ and $C=\bar{B}$. Then $A=C$.

## Proof.

We use a double inclusion argument to prove that $A=C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.
$(C \subseteq A)$ : In order to show that $C \subseteq A$, we show that every element of $C$ is an element of $A$. Namely, $x \in C \Rightarrow x \in A$.

## Definition (Subset)

Let $A$ and $B$ be sets. Then $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

$$
\begin{array}{cccc}
\hline \text { Given } B=\bar{A} \text {, we know } \\
\circ & x \in B \Rightarrow x \in \bar{A} & \circ \\
\circ & x \in \bar{A} \Rightarrow x \in B & \circ \\
\circ & x \notin B \Rightarrow x \notin \bar{A} & \circ \\
\circ & x \notin \bar{A} \Rightarrow & x \notin B & \circ \\
\hline
\end{array}
$$

$$
\begin{array}{cccc}
\text { Given } C=\bar{B}, \text { we know } \\
\circ & x \in C \Rightarrow x \in \bar{B} & \circ \\
\circ & x \in \bar{B} \Rightarrow x \in C & \circ \\
\circ & x \notin C \Rightarrow x \notin \bar{B} & \circ \\
\circ & x \notin \bar{B} \Rightarrow x \notin C & \circ
\end{array}
$$

## Given the set $A$, we know

$\begin{array}{cccc}\circ x \in A & \Rightarrow & x \notin \bar{A} & 0 \\ 0 x \notin \bar{A} & \Rightarrow & x \in A & 0 \\ 0 x \notin A & \Rightarrow & x \in \bar{A} & 0 \\ 0 x \in \bar{A} & \Rightarrow & x \notin A & 0\end{array}$

Given the set $B$, we know - $x \in B \Rightarrow x \notin \bar{B} \quad \circ$

| $\circ x \notin B$ | $\Rightarrow$ | $x \in \bar{B}$ | $\circ$ |
| :--- | :--- | :--- | :--- |
| $\circ$ | $x \in \bar{B}$ | $\Rightarrow$ | $x \notin B$ |

$$
x \in C \Rightarrow
$$

$$
:
$$

$$
\Rightarrow x \in A
$$

| Given $B=\bar{A}$, we know | Given $C=\bar{B}$, we know | Given the set $A$, we know | Given the set $B$, we know |
| :---: | :---: | :---: | :---: |
| - $x \in B \Rightarrow x \in \bar{A} \circ$ | - $x \in C \quad \Rightarrow \quad x \in \bar{B} \quad 0$ | - $x \in A \Rightarrow x \notin \bar{A} \circ$ | - $x \in B \quad \Rightarrow \quad x \notin \bar{B}$ |
| - $x \in \bar{A} \Rightarrow x \in B \quad 0$ | - $x \in \bar{B} \Rightarrow x \in C \quad 0$ | - $x \notin \bar{A} \Rightarrow x \in A \quad 0$ | - $x \notin \bar{B} \Rightarrow x \in B \quad 0$ |
| - $x \notin B \Rightarrow x \notin \bar{A} \circ$ | $\bigcirc x \notin C \Rightarrow x \notin \bar{B} \quad 0$ | $\bigcirc x \notin A \quad \Rightarrow \quad x \in \bar{A} \quad \circ$ | $\bigcirc x \notin B \Rightarrow x \in \bar{B} \quad 0$ |
| - $x \notin \bar{A} \Rightarrow x \notin B \quad \circ$ | $\bigcirc x \notin \bar{B} \Rightarrow x \notin C \quad \circ$ | $\bigcirc x \in \bar{A} \quad \Rightarrow \quad x \notin A \circ$ | - $x \in \bar{B} \Rightarrow x \notin B$ |

$$
x \in C \Rightarrow x \in \bar{B}
$$

(since $C=\bar{B}$ )

$$
\begin{array}{llll}
\text { Given } B=\bar{A}, \text { we know } \\
\circ & x \in B & \Rightarrow x \in \bar{A} & 0 \\
\circ & x \in \bar{A} & \Rightarrow x \in B & 0 \\
\circ & x \notin B & \Rightarrow & x \notin \bar{A}
\end{array} 0
$$

| Given $C=\bar{B}$, we know |  |
| :---: | :---: |
|  | $\bigcirc x \in C \quad x \in \bar{B} \circ$ |
|  | - $x \in \bar{B} \Rightarrow x \in C$ o |
|  | - $x \notin C \Rightarrow x \notin \bar{B} \circ$ |
|  | - $x \notin \bar{B} \Rightarrow x \notin C$ o |

Given the set $A$, we know
$0 \quad x \in A \Rightarrow x \notin \bar{A} \quad 0$
$0 \quad x \notin \bar{A} \Rightarrow x \in A$
0
$0 x \notin A \Rightarrow x \in \bar{A}$
0
0
$x \in \bar{A} \Rightarrow x \notin A$

| Given the set $B$, we know |
| :---: |
| ○ $x \in B \Rightarrow x \notin \bar{B} \circ$ |
| ○ $x \notin \bar{B} \Rightarrow x \in B \circ$ |
| - $x \notin B \Rightarrow x \in \bar{B}$ |
| - $x \in \bar{B} \Rightarrow x \notin B$ ○ |

$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} \\
& \Rightarrow x \notin B \\
& \vdots \\
& \Rightarrow x \in A
\end{aligned}
$$

(since $C=\bar{B}$ )
(by def. of converse of $B$ )

$$
\begin{array}{|llll|}
\hline \text { Given } B=\bar{A}, \text { we know } \\
\circ & x \in B & \Rightarrow x \in \bar{A} & \circ \\
\circ & x \in \bar{A} & \Rightarrow x \in B & 0 \\
\hline \circ & x \notin B & \Rightarrow & x \notin \bar{A}
\end{array} 0
$$


Given the set $A$, we know
0
$x \in A \Rightarrow x \notin \bar{A}$
0
0
$x \notin \bar{A} \Rightarrow x \in A$
0
$x \notin A$
0
$x \in \bar{A} \Rightarrow x \in \bar{A}$ 0

Given the set $B$, we know ○ $x \in B \Rightarrow x \notin \bar{B} \circ$ ○ $x \notin \bar{B} \Rightarrow x \in B \quad 0$ $\circ x \notin B \Rightarrow x \in \bar{B} \quad \circ$
$\circ x \in \bar{B} \Rightarrow x \notin B \quad 0$

$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} \\
& \Rightarrow x \notin B \\
& \Rightarrow x \notin \bar{A}
\end{aligned}
$$

$$
\vdots
$$

$$
\Rightarrow x \in A
$$

(since $C=\bar{B}$ )
(by def. of converse of $B$ )
(since $B=\bar{A}$ )
$\left.\begin{array}{c}\text { Given } B=\bar{A}, \text { we know } \\ 0 \\ x \in B \Rightarrow x \in \bar{A} \\ 0 \\ 0 \\ x \in \bar{A} \Rightarrow x \in B\end{array}\right]$

| Given $C=\bar{B}$, we know |
| :---: |
| 0 |
| $x \in C \Rightarrow x \in \bar{B}$ | 0


| Given the set $A$, we know |  |  |
| :--- | :--- | :--- |
| 0 | $x \in A \Rightarrow x \notin \bar{A}$ | 0 |
| 0 | $x \notin \bar{A} \Rightarrow x \in A$ | 0 |
| 0 | $x \notin A$ | $\Rightarrow$ |
| $0 \in \bar{A}$ | 0 |  |
| 0 | $x \in \bar{A} \Rightarrow x \notin A$ | 0 |


| Given the set $B$, we know |
| :--- |
| $\circ \quad x \in B \Rightarrow x \notin \bar{B} \quad \circ$ |
| $0 \quad x \notin \bar{B} \Rightarrow x \in B \quad 0$ |
| $\circ \quad x \notin B \Rightarrow x \in \bar{B} \quad 0$ |
| $\circ \quad x \in \bar{B} \Rightarrow x \notin B \quad 0$ |

$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} \\
& \Rightarrow x \notin B \\
& \Rightarrow x \notin \bar{A} \\
& \Rightarrow x \in A
\end{aligned}
$$

(since $C=\bar{B}$ )
(by def. of converse of $B$ )
(since $B=\bar{A}$ )
(by def. of converse of $A$ )

## Proof.

We use a double inclusion argument to prove that $A=C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.
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Suppose $x \in C$. Then

$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} & & (\text { since } C=\bar{B}) \\
& \Rightarrow x \notin B & & \text { (by def. of converse of } B) \\
& \Rightarrow x \notin \bar{A} & & (\text { since } B=\bar{A}) \\
& \Rightarrow x \in A & & \text { (by def. of converse of } A) .
\end{aligned}
$$

## Proof.

We use a double inclusion argument to prove that $A=C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.
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$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} & & (\text { since } C=\bar{B}) \\
& \Rightarrow x \notin B & & \text { (by def. of converse of } B) \\
& \Rightarrow x \notin \bar{A} & & (\text { since } B=\bar{A}) \\
& \Rightarrow x \in A & & \text { (by def. of converse of } A) .
\end{aligned}
$$

Thus $C \subseteq A$ by the definition of subset.

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$$
\begin{aligned}
x \in C & \Rightarrow x \in \bar{B} & & (\text { since } C=\bar{B}) \\
& \Rightarrow x \notin B & & \text { (by def. of converse of } B) \\
& \Rightarrow x \notin \bar{A} & & (\text { since } B=\bar{A}) \\
& \Rightarrow x \in A & & \text { (by def. of converse of } A) .
\end{aligned}
$$

Thus $C \subseteq A$ by the definition of subset.
$(A \subseteq C):$ [Exercise.]
Since $C \subseteq A$ and $A \subseteq C$, we conclude that $A=C$.

