Proof Portfolio

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January 11, 2017

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Preface

Keys to mathematical writing

1. **Guide the reader.** As with any piece of writing, it is important to provide the reader with context. The first line(s) of a proof should outline what will be proved and how the proof will be carried out. Longer proofs may have multiple interludes to remind the reader

   (a) what you are trying to accomplish,

   (b) what parts of the proof have been completed so far, and

   (c) what part of the proof will be tackled next.

2. **Write in complete English sentences.** Every statement should be a sentence. Sentences should be organized into paragraphs. The rules of spelling, punctuation, and grammar apply to mathematics as well.

3. **Be precise.** Avoid using (read: do not use) ambiguous words and phrases. Wherever possible, be explicit about the objects to which you are referring.

4. **Define all of your notation.** The first time you use a symbol, state explicitly what that symbol means (even if the symbol previously appeared in the statement of a problem, theorem, or definition).

5. **Use appropriate symbols.** Mathematical symbols should match the context in which they are used. Mathematical phrases should integrate seamlessly into the surrounding text. It is a faux pas to begin a sentence with a mathematical symbol. *Be careful not to use the same symbol to represent multiple objects.*

6. **Justify your claims.** For the most part, each sentence in the body of your proof should contain two statements:

   (a) a statement of fact (usually) a logical consequence of the preceding statement, and

   (b) justification for why the logical statement is true.

   Usually statements are justified by citing a definition or a theorem, or by providing simple algebraic steps. In some cases, more than one line of justification is needed.

7. **Write for your peers.** Write arguments that can be understood by your classmates. Keep your statements simple, and strive for clarity. Use words that everyone can understand.

8. **Write the complete statement of each definition.** Write out the precise statement of each definition you use in your proof. (You may want to do this before your proof, rather than within the proof, to avoid filling the proof with interjections.) Writing out the definitions will help you learn the statements and also provide a helpful reminder to the reader.

9. **Proofread.** In fact, *read your proof out loud.* Do your statements read smoothly, or are there gaps? Do you find the need to insert words/phrases/pauses/interludes/etc. to make sense of or clarify your statements? Your readers should not have to guess to fill in blanks.

Proof 1

Prove that if $A = B$, then $A \subseteq B$ and $B \subseteq A$.

1.1 First draft

Due Monday, February 8 at 6:00 PM.

Proof. Write your proof here. $\square$

1.2 Second draft

Proof. After receiving my comments, write your revised proof here. (Do not make changes in the first section.) $\square$

Proof 2

Prove that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Remark. Combined with Proof 1, these two statements yield the following theorem.

Theorem 2.1 (Double containment)**.** *If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

The "if and only if" means that the statement can be read in either direction. Specifically, "$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$" is equivalent to the following statements:

(a) if $A = B$, then $A \subseteq B$ and $B \subseteq A$, and

(b) if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

This result (in particular, statement (b)) is one of the most common methods for showing that two sets are equal. Since this result relies on showing that each of the two sets is contained within the other, we call this result the Double Containment (or Double Inclusion) Principle.

2.1 First draft

Due Wednesday, February 10 at 6:00 PM.

Proof. Write your proof here. $\square$

Proof 3

Prove that if $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

3.1 First draft

Due Monday, February 15 at 6:00 PM.

Proof. $\square$

Proof 4

Prove that $\{x \in \mathbb{Z} : 55 \mid x\} \subseteq \{x \in \mathbb{Z} : 11 \mid x\}$.

4.1 First draft

Due Friday, February 19 at 6:00 PM.

Proof. $\square$

Proof 5

Prove that $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

5.1 First draft

Due Wednesday, February 24 at 6:00 PM.

Proof. $\square$

Proof 6

Prove that $\{6n + 9 : n \in \mathbb{Z}\} = \{6n - 3 : n \in \mathbb{Z}\}$.

Remark. To prove that two sets are equal, give a double containment argument.

6.1 First draft

Due Monday, February 29 at 6:00 PM.

Proof. $\square$

Proof 7

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Remark. To prove that two sets are equal, give a double containment argument.

7.1 First draft

Due Friday, March 4 at 6:00 PM.

Proof. $\square$

Proof 8

Prove that $\displaystyle\bigcap_{y>0,\, y\in\mathbb{R}} A_y = \underline{\hspace{3cm}}$,

where $A_y = (-y, y) \subseteq \mathbb{R}$ is an open interval. It is up to you to determine the set on the right and to prove the equality. If possible, determine the set on the right explicitly.

8.1 First draft

Due Wednesday, March 9 at 6:00 PM.

Proof. $\square$

 $\bigcap_{Y>0,\, Y\in\mathbb{R}} A_Y =$ ________________.

Proof 9

Suppose x, y, and $z \in \mathbb{Z}$, and $x \neq 0$. Prove that if $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Remark. Give a proof by contrapositive.

9.1 First draft

Due Monday, March 14 at 6:00 PM.

Proof. $\square$

Proof 10

Prove that $\sqrt{6}$ is irrational.

Remark. Give a proof by contradiction.

10.1 First draft

Due Wednesday, March 16 at 6:00 PM.

Proof. $\square$

Proof 11

Prove that $\mathcal{B} = \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$.

11.1 First draft

Due Wednesday, March 30 at 6:00 PM.

Proof. $\square$

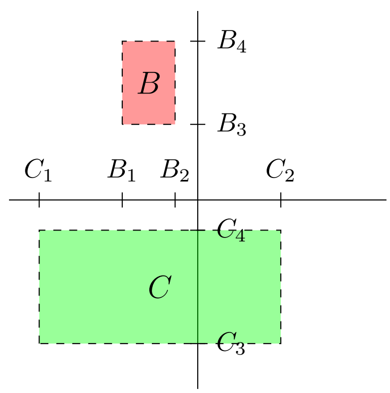
Proof 12

Determine if these sets are open and/or closed.

Let $\mathcal{B}$ be the set of "open rectangles" in $\mathbb{R}$:

$$\mathcal{B} = \{(a, b) \times (c, d) \subseteq \mathbb{R}^2 : (a, b) \subseteq \mathbb{R}, (c, d) \subseteq \mathbb{R}\}.$$

Here are two examples of elements in $\mathcal{B}$: $B = (B_1, B_2) \times (B_3, B_4)$ and $C = (C_1, C_2) \times (C_3, C_4)$.



The set $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$. (You do not have to prove that $\mathcal{B}$ is a basis.) Consider the following sets.

1. The interior of the unit circle: $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
2. The integer lattice $\mathbb{Z}^2$: $U = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

For each set U, complete the following.

a. Determine if U is open or not open. Prove your claim.

b. Determine if U is closed or not closed. Prove your claim.

In the end, you should have four short proofs.

12.1 First draft

Due Monday, April 4 at 6:00 PM.

1. The interior of the unit circle: $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

   (a) Claim: U is (pick one: open/not open).

   Proof. $\square$

   (b) Claim: U is (pick one: closed/not closed).

   Proof. $\square$

2. The integer lattice $\mathbb{Z}^2$: $U = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

   (a) Claim: U is (pick one: open/not open).

   Proof. $\square$

   (b) Claim: U is (pick one: closed/not closed).

   Proof. $\square$

Proof 13

Prove that the intersection of two open sets is open.

13.1 First draft

Due Friday, April 8 at 6:00 PM.

Proof. □

Proof 14

Prove that R is an equivalence relation on $\mathbb{Z}$.

Let $A = \mathbb{Z}^2 - \{(0,0)\}$, and let R be the relation on A defined by

$$R = \{((a,b),(c,d)) : ad - bc = 0\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.

14.1 First draft

Due Friday, April 8 at 6:00 PM.

Proof. $\square$

Proof 15

Prove the following correspondence between equivalence relations and partitions.

Let A be a set.

1. Prove that if R is an equivalence relation on A, then $\{[a] : a \in A\}$ is a partition of A.
2. Prove that if P is a partition of A, then $R = \{(x, y) : x, y \in X \text{ and } X \in P\}$ is an equivalence relation on R.

15.1 First draft

Due Wednesday, April 20 at 6:00 PM.

Proof. $\square$