MATH 2001
FUNCTIONS

Due Wednesday, April 20.

Book exercises: Section 11.4: 6, 7.

Proofs: Final draft of Proof 14 and first draft of Proof 15.

Due Wednesday, April 27.

Book exercises: Section 12.1: 2, 5, 8, 12. Section 12.2: 8, 14.

Definition. Let A and B be sets. We say that R is a *relation from A to B* if $R \subseteq A \times B$.

Definition. Let A and B be sets, and let f be a relation from A to B. The relation f is a *function from A to B* (written $f\colon A \to B$) if for each $a \in A$, the relation f contains exactly one element of the form (a, b).

Since (a, b) is unique to a, we write $f(a) = b$.

Definition. Let $f\colon A \to B$. The *domain* of f is A, and the *codomain* of f is B. The *image* (or *range*) of f is the set

$$\mathrm{im}(f) = \{b \in B : (a, b) \in f\}.$$

Definition. A function $f\colon A \to B$ is *injective* (or *one-to-one*) if whenever $f(a) = f(b)$, then $a = b$. Equivalently, if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$.

Definition. A function $f\colon A \to B$ is *surjective* (or *onto*) if $\mathrm{im}(f) = B$. That is, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function f is *bijective* if f is both injective and surjective.

Exercise 1. Give an example of a function that is

a.) injective but not surjective;

b.) surjective but not injective;

c.) neither injective nor surjective;

d.) bijective.

Exercise 2. Prove that $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x$ is injective but not surjective.

Exercise 3. Prove that $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $f(x) = |x|$ is surjective but not injective. (Here $\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} : x \geq 0\}$.)

Exercise 4. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are injective functions, then $g \circ f\colon A \to C$ is injective.

Exercise 5. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are surjective functions, then $g \circ f\colon A \to C$ is surjective.