Definition

Linear Combination

(Sets of) Vectors

Definition

Linearly Independent/dependent

(Sets of) Vectors

Definition

Span

(Sets of) Vectors

Definition

Inverse/Invertible Matrix

Matrices

Definition

Matrix Transpose

Matrices

Definition

Elementary Row Operation

Matrices

Definition

Row equivalent

Matrices

Definition

Reduced Row Echelon Form

Matrices

Definition

Pivot (position)

Matrices

Definition

Pivot/Non-pivot column

Matrices

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is *linearly independent* if the only solution to

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

is the trivial solution: $x_1 = \cdots = x_n = 0$. If there is more than one solution to this equation, then the set of vectors is *linearly dependent*.

A *linear combination* of a vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a sum of scalar multiples of the vectors:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n$$

(the x_i's are scalars).

An $n \times n$ matrix A is *invertible* if there exists another $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.

The *span* of a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (denoted $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$) is the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

There are three types of *elementary row operations* for matrices.

1. Swapping two rows of the matrix.
2. Adding a multiple of one row to another row.
3. Scaling a row by a non-zero constant.

If A is an $m \times n$ matrix, then the *transpose* of A (denoted A^T) is an $n \times m$ matrix whose i-th row is the i-th column of A:

$$\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix}^T = \begin{bmatrix} \text{—} & \mathbf{v}_1^T & \text{—} \\ & \vdots & \\ \text{—} & \mathbf{v}_n^T & \text{—} \end{bmatrix}.$$

The *reduced row echelon form* of an $m \times n$ matrix A (denoted rref A) is the $m \times n$ matrix which is row equivalent to A and satisfies the following.

1. The leading non-zero entry of each row is 1.
2. All other entries in the columns with leading 1's are 0's.
3. The leading 1 in any row is to the right of all leading 1's in the rows above. Rows of 0's are at the bottom of the matrix.

Two $m \times n$ matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

A column of a matrix is a *pivot column* if the column contains a pivot. Otherwise the column is a *non-pivot column*.

A *pivot position* (or simply, *pivot*) in a matrix A is a position corresponding to a leading 1 in rref A.

Definition

Column Space

Matrices

Definition

Rank

Matrices

Definition

Null Space

Matrices

Definition

Nullity

Matrices

Definition

Row space

Matrices

Definition

Identity Matrix

Matrices

Definition

Identity Matrix

Matrices

Definition

Vector Space

Vector Spaces

Definition

Subspace

Vector Spaces

Definition

Basis

Vector Spaces

The *rank* of a matrix A (denoted $\operatorname{rank} A$) is the dimension of the column space of the matrix, which is the number of pivots/pivot columns in A.

The *column space* of a matrix A (denoted $\operatorname{Col} A$) is the span of the columns of A.

The *nullity* of a matrix A is the dimension of the null space of A, which is the number of non-pivot columns in A.

The *null space* of a matrix A (denoted $\operatorname{Nul} A$) is the set of all vectors $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{0}$.

A *diagonal matrix* is an $n \times n$ matrix where all the entries off the diagonal are 0's.

The *row space* of a matrix A is the column space of A^T.

A *vector space* is a set of objects (e.g. vectors) along with two operations, vector addition and scalar multiplication, which satisfy ten axioms (not listed here).

The $n \times n$ *identity matrix* is a diagonal matrix with 1's on its diagonal.

A *basis* for a vector space V is a set of linearly independent vectors that spans V (i.e. the span of the vectors is V).

Let V be a vector space. A subspace H of V is a subset of V that satisfies

1. $\mathbf{0} \in H$;
2. H is closed under vector addition (if $\mathbf{u}, \mathbf{v} \in H$, then $\mathbf{u} + \mathbf{v} \in H$);
3. H is closed under scalar multiplication (if $\mathbf{u} \in H$ and k is a scalar, then $k\mathbf{u} \in H$).

Every subspace is a vector space.

Definition

Standard Basis for $\mathbb{R}^n$

Vector Spaces

Definition

Dimension

Vector Spaces

Definition

Domain/Codomain

Functions and Linear Transformations

Definition

Image (range)

Functions and Linear Transformations

Definition

Kernel

Functions and Linear Transformations

Definition

Onto

Functions and Linear Transformations

Definition

One-to-one

Functions and Linear Transformations

Definition

Bijection and Inverse

Functions and Linear Transformations

Definition

Linear Transformation

Functions and Linear Transformations

Definition

Standard Matrix for a Linear Transformation

Functions and Linear Transformations

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then the *dimension* of V is n, the number of vectors in the basis.
(A vector space may also be infinite dimensional; it may not be possible to write down a basis for the vector space in this case.)

The *standard basis* for $\mathbb{R}^n$ is the set of columns of the $n \times n$ identity matrix. (The i-th standard basis vector is denoted $\mathbf{e}_i$.)

If $T\colon V \to W$ is a function, then the *image* (or *range*) of T (denoted $\operatorname{im} T$) is the set of all outputs of T. That is, the image of T is the set of all vectors $\mathbf{b}$ for which $T(\mathbf{x}) = \mathbf{b}$ has a solution.
If A is the standard matrix for T, then the image of T is the column space of A, which is a subspace of W.

If $T\colon V \to W$ is a function, then V is the *domain* of T, and W is the *codomain* of T.

If $T\colon V \to W$ is a function, then T is *onto* if $T(\mathbf{x}) = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in W$. In this case, the image of T is W.

If $T\colon V \to W$ is a function, then the *kernel* of T is the set of all solutions to $T(\mathbf{x}) = \mathbf{0}$.
If A is the standard matrix for T, then the kernel of T is the null space of A, which is a subspace of V.

If $T\colon V \to W$ is a function that is both one-to-one and onto, then T is a *bijection*. Moreover, there exists a function $T^{-1}\colon W \to V$ satisfying $T^{-1}(T(\mathbf{x})) = \mathbf{x}$ and $T(T^{-1}(\mathbf{b})) = \mathbf{b}$ for all $x \in V$ and $\mathbf{b} \in W$. The function T^{-1} is the *inverse* of T.

If $T\colon V \to W$ is a function, then T is *one-to-one* if $T(\mathbf{x}) = \mathbf{b}$ has at most one solution for each $\mathbf{b} \in W$.

If $T\colon V \to W$ is a linear transformation, $\dim V = m$ and $\dim W = n$, then there is an $n \times m$ matrix A that satisfies $T(\mathbf{x}) = A\mathbf{x}$. This matrix A is the *standard matrix* for T. Moreover, the columns of A are the images of the standard basis vectors under T:

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \\ | & | & & | \end{bmatrix}.$$

Let V and W be vector spaces. A function $T\colon V \to W$ is a *linear transformation* if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

for any $\mathbf{u}, \mathbf{v} \in V$ and any scalar k.

Definition

Coordinates

Coordinate Systems

Definition

$\mathfrak{B}$-matrix for a linear transformation

Coordinate Systems

Definition

Determinant

Matrix Invariant

Theorem

Eigenvalue formula for determinant

Matrix Invariant

Definition

Eigenvalue

Eigenvalues and Eigenvectors

Definition

Change of basis matrix

Coordinate Systems

Definition

Trace

Matrix Invariant

Theorem

Eigenvalue formula for trace

Matrix Invariant

Definition

Characteristic polynomial

Eigenvalues and Eigenvectors

Definition

Multiplicity of an eigenvalue

Eigenvalues and Eigenvectors

Let $\mathfrak{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathfrak{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_n\}$ be bases for a vector space V. Then the *change of basis matrix* from $\mathfrak{B}$ to $\mathfrak{D}$ is

$$\underset{\mathfrak{D}\leftarrow\mathfrak{B}}{P} = \begin{bmatrix} | & | & & | \\ [\mathbf{b}_1]_{\mathfrak{D}} & [\mathbf{b}_2]_{\mathfrak{D}} & \cdots & [\mathbf{b}_n]_{\mathfrak{D}} \\ | & | & & | \end{bmatrix}.$$

For any vector $\mathbf{v} \in V$, this matrix satisfies

$$\underset{\mathfrak{D}\leftarrow\mathfrak{B}}{P}[\mathbf{v}]_{\mathfrak{B}} = [\mathbf{v}]_{\mathfrak{D}}.$$

If $\mathfrak{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V, then each $\mathbf{b} \in V$ is expressible as a unique linear combination of the basis elements:

$$\mathbf{b} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n.$$

The *coordinates* for $\mathbf{b}$ are the coefficients in this expansion, and we write

$$[\mathbf{b}]_{\mathfrak{B}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T.$$

The *trace* of a square matrix is the sum of the entries on the diagonal of the matrix.

Let $T\colon V \to W$ be a linear transformation between vector spaces. Let $\mathfrak{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V and $\mathfrak{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_n\}$ be a basis for W. Then the $\mathfrak{B}$-*matrix* for the linear transformation is

$$B = \begin{bmatrix} | & | & & | \\ [T(\mathbf{b}_1)]_{\mathfrak{D}} & [T(\mathbf{b}_2)]_{\mathfrak{D}} & \cdots & [T(\mathbf{b}_n)]_{\mathfrak{D}} \\ | & | & & | \end{bmatrix}.$$

For any square matrix, the trace of the matrix is equal to the sum of its eigenvalues.

The *determinant* of a square matrix is the sum of the entries on the diagonal of the matrix.

Let A be a square matrix. The *characteristic polynomial* of A is $\det(A - \lambda I)$.

For any square matrix, the determinant of the matrix is equal to the product of its eigenvalues.

Let A be an $n \times n$ matrix, and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of A. Then the characteristic polynomial of A factors into

$$\det(A - \lambda I) = (\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_k)^{\alpha_k}.$$

The *multiplicity* of the eigenvalue λ_i is α_i. Necessarily, the sum of the multiplicities equals n: $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. Thus every $n \times n$ matrix has n eigenvalues when counted with multiplicity.

Let A be a square matrix. The *eigenvalues* of A are the roots of the characteristic polynomial of A.

Definition

Eigenvector

Eigenvalues and Eigenvectors

Definition

Eigenspace

Eigenvalues and Eigenvectors

Theorem

Bounds for the dimension of an eigenspace

Eigenvalues and Eigenvectors

Definition

Eigenbasis

Eigenvalues and Eigenvectors

Definition

Diagonalizable

Eigenvalues and Eigenvectors

Definition

Rotation-scaling matrix

Eigenvalues and Eigenvectors

Let A be a square matrix, and let λ be an eigenvalue for A. The *eigenspace* corresponding to λ (denoted E_λ) is

$$E_\lambda = \text{Nul}(A - \lambda I).$$

The eigenspace E_λ is the set of all eigenvalues corresponding to λ.

Let A be a square matrix, and let λ be an eigenvalue for A. A vector $\mathbf{v}$ is an *eigenvector* corresponding to the eigenvalue λ if $\mathbf{v}$ is a non-zero vector that satisfies $A\mathbf{v} = \lambda\mathbf{v}$.

Let A be a square matrix. An *eigenbasis* for A is a basis for $\text{Col}\,A$ that consists entirely of eigenvectors of A.
The matrix A has an eigenbasis if and only if $\dim E_\lambda = \text{multiplicity}(\lambda)$ for each eigenvalue of A.

Let A be a square matrix, and let λ be an eigenvalue for A. Then

$$1 \leq \dim E_\lambda \leq \text{multiplicity}(\lambda).$$

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a - bi$ and corresponding eigenvector $\mathbf{a} + \mathbf{b}i$. Then the matrices

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad Q = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}$$

satisfy $AQ = QC$. The matrix C is the *rotation-scaling* matrix for A.

A square matrix A is *diagonalizable* if there exist a diagonal matrix D and an invertible matrix P such that $AP = PD$.
Moreover, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A with respective eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an eigenbasis for A, then P is the matrix with the eigenvectors as columns, and D is the matrix with the eigenvalues on its diagonal.