1.2 — Matrices, vectors, and Gauss-Jordan elimination
University of Massachusetts Amherst
Math 235 — Spring 2014

A matrix is a rectangular array of numbers. We can be more specific by giving the dimensions of the matrix, that is, an $n \times m$ *matrix* has n rows and m columns.

This is a 2×3 matrix: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, and this is a 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Definition 1. A *square matrix* has the same number of rows and columns, and the (main) *diagonal of a square matrix* are the elements $a_{11}, a_{22}, \ldots, a_{nn}$.

A square matrix is *upper triangular* if every entry below the diagonal is 0.

A square matrix is *lower triangular* if every entry above the diagonal is 0.

A square matrix is a *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. That is, the only non-zero entries of a diagonal matrix are on the matrix's diagonal. Diagonal matrices are both upper triangular and lower triangular.

Example 2. Identify the square, diagonal, upper triangular, and lower triangular matrices.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A vector is a matrix with only one row or one column, which we will often specify by saying *row vector* for a matrix with one row, and *column vector* for a matrix with one column. The standard convention is to write vectors in column form, so unless otherwise specified, "vector" will mean "column vector".

Recall: There is a one-to-one correspondence between vectors and points in space.

The rules of matrix multiplication (which we will get to in Section 1.3) allow us to reinterpret systems of linear equations as a matrix product.

The system $\left\{ \begin{array}{r} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{array} \right\}$ is equivalent to the matrix product $\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

The matrix $\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$ is called the *coefficient matrix* of the system. For solving linear systems, we work with the *augmented matrix* $\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix}$.

Example 3. Solve the system above using the augmented matrix.

Example 4. Find all solutions to the following system of equations.

$$\left\{\begin{array}{rcrcrcrcrcl} 2x_1 &+& 4x_2 &-& 2x_3 &+& 2x_4 &+& 4x_5 &=& 2 \\ x_1 &+& 2x_2 &-& x_3 &+& 2x_4 & & &=& 4 \\ 3x_1 &+& 6x_2 &-& 2x_3 &+& x_4 &+& 9x_5 &=& 1 \\ 5x_1 &+& 10x_2 &-& 4x_3 &+& 5x_4 &+& 9x_5 &=& 9 \end{array}\right\}$$

Definition 5. A matrix is in *reduced row-echelon form* if it satisfies all of the following conditions:

(1) If a row has nonzero entries, then the first nonzero entry is a 1, called a *leading* 1, or *pivot*.
(2) If a column contains a leading 1, then all the other entries in that column are 0.
(3) If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

We denote the reduced row-echelon form of a matrix A by $\text{rref}(A)$.

The operations that we use to reduce these matrices into row-echelon form are called *elementary row operations*.

Definition 6. The elementary row operations are

- Multiply (or divide) a row by a non-zero scalar.
- Subtract a multiple of one row from another row.
- Swap two rows.

ADDITIONAL EXERCISES

(1) Find all solutions to the systems

(a) $\left\{ \begin{array}{r} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{array} \right\}$ (b) $\left\{ \begin{array}{r} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{array} \right\}$

(2) Find the polynomial of degree 3 (e.g. $f(t) = at^3 + bt^2 + ct + d$) whose graph passes through the points $(0, 1), (1, 0), (-1, 0)$, and $(2, -15)$.

(3) Consider the system of equations

$$\left\{ \begin{array}{r} y + 2kz = 0 \\ x + 2y + 6z = 2 \\ kx + 2z = 1 \end{array} \right\},$$

where k is an arbitrary constant.
(a) For which values of the constant k does this system have a unique solution?
(b) When is there no solution?
(c) When are there infinitely many solutions?

(4) Consider the chemical reaction

$$a\,\mathrm{NO_2} + b\,\mathrm{H_2O} \to c\,\mathrm{HNO_2} + d\,\mathrm{HNO_3},$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, the number of oxygen atoms must satisfy

$$2a + b = 2c + 3d.$$

Find the smallest positive integers that balance the chemical reaction.

(5) Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}? \qquad \text{Explain.}$$