1.3 — On the solutions of linear systems; matrix algebra
University of Massachusetts Amherst
Math 235 — Spring 2014

Example 1. The reduce row-echelon forms for three systems are given below. How many solutions are there in each case.

a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Theorem 2. *A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row* $[0 \quad 0 \quad \cdots \quad 0 \quad 1]$*, representing the equation* $0 = 1$*. If a linear system is consistent, then it either has exactly one solution, or infinitely many solutions.*

Definition 3. The *rank* of a matrix A is the number of leading 1's in $\text{rref}(A)$.

Example 4. Compute the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 0 \end{bmatrix}$.

Example 5. Let A be the coefficient matrix of a system of n linear equations with m variables

a) What are the dimensions of A?
b) Verify that $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
c) Show that if $\text{rank}(A) = m$, then the system has at most one solution.
d) Show that if $\text{rank}(A) = n$, then the system has no solutions or infinitely many solutions.

Theorem 6. *A linear system with fewer equations than unknowns has either no solutions or infinitely many solutions. In other words, if a system has a unique solution, then it must have at least as many equations as variables.*

Theorem 7. *A linear system of n equations in n variables has a unique solution if and only if the rank of its coefficient matrix A is n. In this case,* $\mathrm{rref}(A)$ *is a diagonal matrix with 1's on its diagonal (also known as an identity matrix).*

MATRIX ALGEBRA

Matrix addition:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix} =$$

Scalar multiplication:

$$4\begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & 0 \end{bmatrix} =$$

Dot product:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} =$$

Matrix multiplication:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ -1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} =$$

LINEAR COMBINATIONS

Definition 8. A vector $\vec{b}$ is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ if there exist scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots x_m\vec{v}_m.$$

Theorem 9. *If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x}$ is a vector with components $x_1, \ldots, x_m$, then*

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

That is, the vector $A\vec{x}$ is a linear combination of the columns vectors in A.

Example 10. Find a 3×3 matrix A such that $A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $A\begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}4\\6\\8\end{bmatrix}$, and $A\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

ADDITIONAL EXERCISES

(1) Prove Theorem 9 in the 2×2 case. That is, show that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

(2) Let $\vec{x} = \begin{bmatrix}5\\3\\-9\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}2\\0\\1\end{bmatrix}$.
  (a) Find a diagonal matrix A such that $A\vec{x} = \vec{y}$.
  (b) Find a matrix A of rank 1 such that $A\vec{x} = \vec{y}$.
  (c) Find a matrix with all nonzero entries such that $A\vec{x} = \vec{y}$.

(3) Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix}1 & 2 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}2\\1\\0\end{bmatrix}$