3.2 — Bases and linear independence
University of Massachusetts Amherst
Math 235 — Spring 2014

Definition 1. A subset W of $\mathbb{R}^n$ is called a *linear subspace* of $\mathbb{R}^n$ if it satisfies the following three properties:
(a) W contains the zero vector in $\mathbb{R}^n$
(b) W is closed under addition: If $\vec{w}_1$ and $\vec{w}_2$ are in W, then so is $\vec{w}_1 + \vec{w}_2$.
(c) W is closed under scalar multiplication: if $\vec{w}$ is in W, then $k\vec{w}$ is in W for any scalar k.

Theorem 2. *If $T(\vec{x}) = A\vec{x}$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then*
(a) $\ker(T) = \ker(A)$ *is a linear subspace of* $\mathbb{R}^m$*, and*
(b) $\operatorname{im}(T) = \operatorname{im}(A)$ *is a linear subspace of* $\mathbb{R}^n$*.*

Proof.

□

Example 3. Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \colon x \geq 0, y \geq 0 \right\}$ a linear subspace of $\mathbb{R}^2$?

Example 4. Classify all the linear subspaces of $\mathbb{R}^3$.

Recall the definition of the span of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$.

Example 5. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Find vectors in $\mathbb{R}^3$ that span the image of A. What is the smallest number of vectors needed to span the image of A?

Definition 6. Consider the set of vectors $\{\vec{v}_1, \dots, \vec{v}_m\} \subset \mathbb{R}^n$.

(a) We say that a vector $\vec{v}_i$ is *redundant* if $\vec{v}_i$ is a linear combination of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{i-1}$.

(b) The vectors $\vec{v}_1, \dots, \vec{v}_m$ are called *linearly independent* if none of them are redundant. Otherwise the vectors are called *linearly dependent.*

(c) We say that the vectors $\vec{v}_1, \dots, \vec{v}_m$ form a *basis* of a subspace V of $\mathbb{R}^n$ if they span V and are linearly independent.

Definition 7. The *standard* basis for $\mathbb{R}^n$ is the set of vectors $\{\vec{e}_1, \dots, \vec{e}_n\}$, where $\vec{e}_i$ has a 1 in the i-th component, and 0's everywhere else.

The standard basis for $\mathbb{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. The standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. However, any set of n linearly in dependent vectors is a basis for $\mathbb{R}^n$.

Theorem 8. *Suppose that the set of vectors* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *spans* V*, then the set of vectors* $\{\vec{v}_1, \dots, \vec{v}_m\}$ *contains a basis for* V*.*

Theorem 9. *The pivot columns of a matrix* A *are a basis for* $\operatorname{im}(A)$*.*

Example 10. Find a basis for $\operatorname{im}(A)$, where $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Example 11. Does $\operatorname{im}(A) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Example 12. Which sets of vectors contains a basis for $\mathbb{R}^3$?

$$(a) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad (b) \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (c) \quad \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Example 13. Find a basis for the kernel of the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}$.

Theorem 14. *The vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent if and only if the only solution to*

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m = \vec{0}$$

is the trivial solution: $a_1 = a_2 = \cdots = a_m = 0$.

Example 15. Suppose that A is an $n \times m$ matrix whose columns are linearly independent. What is $\ker(A)$?

Theorem 16. *Let $\{\vec{v}_1, \dots, \vec{v}_m\}$ be a set of vectors in $\mathbb{R}^n$. The following are equivalent.*
(a) The vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.
(b) None of the vectors $\vec{v}_1, \dots, \vec{v}_m$ are redundant.
(c) None of the vectors is a linear combination of the other vectors.
(d) If $\quad a_1\vec{v}_1 + \cdots + a_m\vec{v}_m = \vec{0}, \quad$ *then* $\quad a_1 = \cdots = a_m = 0.$
(e) $\ker \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \{\vec{0}\}.$
(f) $\operatorname{rank} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = m.$

Theorem 17. *Let $\{\vec{v}_1, \dots, \vec{v}_m\}$ be a set of vectors in a subspace V of $\mathbb{R}^n$. The set of vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent if and only if every vector $\vec{v} \in V$ can be expressed uniquely as a linear combination*

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m.$$

Proof.

□

ADDITIONAL EXERCISES

(1) Find a nontrivial relation among the vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$.

(2) Is the set of vectors $\left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x \leq y \leq z\right\}$ a linear subspace of $\mathbb{R}^3$? Explain.

(3) Let $A = \begin{bmatrix}| & | & | & |\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4\\ | & | & | & |\end{bmatrix}$ and suppose $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in the kernel of A. Express $\vec{v}_4$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(4) Suppose that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent vectors in $\mathbb{R}^4$. Find rref $\begin{bmatrix}| & | & |\\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3\\ | & | & |\end{bmatrix}$.