3.3 — Bases and linear independence
University of Massachusetts Amherst
Math 235 — Spring 2014

Recall the definition for the basis of a linear subspace V of $\mathbb{R}^n$.

Theorem 1. *All bases of a linear subspace V of $\mathbb{R}^n$ consists of the same number of vectors.*

Definition 2. The *dimension* of a linear subspace V of $\mathbb{R}^n$ is the number of vectors in any basis of V. We denote the dimension of V by $\dim(V)$.

Example 3. The dimension of $\mathbb{R}^n$ is n because

Example 4. The dimension of the trivial space $\{\vec{0}\}$ is

Theorem 5. *Let V be an m-dimensional linear subspace of $\mathbb{R}^n$.*
(a) We can find at most m linearly independent vectors in V.
(b) We need at least m vectors to span V.
(c) If m vectors in V are linearly independent, then they form a basis of V.
(d) If m vectors in V span V, then they a form a basis of V.
(e) Any linearly independent set can be expanded to a basis.
(f) Any spanning set can be shrunk to a basis.

Example 6. Find a basis and determine the dimensions of the image and kernel of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}.$$

Theorem 7 (Rank-Nullity Theorem)**.** *For any $n \times m$ matrix A, we have*

$$\dim(\ker(A)) + \dim(\operatorname{im}(A)) = m.$$

Proof.

□

Theorem 8. *Let A be an $n \times n$ matrix. The following are equivalent.*
(a) A is invertible.
(b) The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for each $\vec{b} \in \mathbb{R}^n$.
(c) $\operatorname{rref}(A) = I_n$.
(d) $\operatorname{rank}(A) = n$.
(e) $\operatorname{im}(A) = \mathbb{R}^n$.
(f) $\ker(A) = \{\vec{0}\}$.
(g) The columns of A form a basis of $\mathbb{R}^n$.
(h) The columns of A are linearly independent.
(i) The columns of A span $\mathbb{R}^n$.

ADDITIONAL EXERCISES

(1) Find a basis for the image and kernel of each matrix.

$$(a)\ \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

(2) Can you find a 3×3 matrix A such that $\operatorname{im}(A) = \ker(A)$? Explain.
(3) Find a basis for, and the dimension of, the solution set of the system

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 2x_4 = 0.$$

(4) Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Find a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that $\ker(T) = \{\vec{0}\}$ and $\operatorname{im}(T) = V$. Describe T by its matrix A.