4.2 — Linear Transformations
University of Massachusetts Amherst
Math 235 — Spring 2014

Recall that a vector space is a set of objects that is closed under two operations: addition and scalar multiplication.

Definition 1. A *linear differential equation* with constant coefficients is an equation of the form

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_1 f^{(1)}(x) + a_0 f(x) = 0,$$

where $f^{(i)}(x)$ is the i-th derivative of $f(x)$, and $a_0, \ldots, a_{n-1}$ are constants.

Theorem 2. *The solutions of the differential equation*

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_1 f^{(1)}(x) + a_0 f(x) = 0,$$

form an n-dimensional subspace of C^∞ (the set of all infinitely differentiable functions).

Example 3. Find all solutions of the differential equation

$$f''(x) + f'(x) - 6f(x) = 0.$$

Definition 4. Let V and W be vector spaces. A function T from V to W is called a *linear transformation* if

$$T(f+g) = T(f) + T(g) \quad \text{and} \quad T(kf) = kT(f)$$

for all elements f, g of V and for all scalars k. For any linear transformation, we have similar concepts for *image*, *kernel*, *rank*, and *nullity*.

Example 5. Show that differentiation is a linear transformation from C^∞ (the space of infinitely differentiable functions) to itself.

Example 6. Let I be the function defined by $I(f) = \int_0^1 f(x)\,dx$. Show that I is a linear transformations from $C[0,1]$ (the set of continuous functions on the interval $[0,1]$) to $\mathbb{R}$. What is the rank of I?

Example 7. Let V be the set of infinite sequences of real numbers, and let T be the shift map:

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

Show that T is a linear transformation from V to V. What is the kernel of T? What is the image of T?

Definition 8. An invertible linear transformation T is called an *isomorphism*. We say that two vector spaces V and W are *isomorphic* if there exists an isomorphism from V to W (or vice versa).

Example 9. Show that $\mathbb{R}^{2\times 2}$ is isomorphic to $\mathbb{R}^4$.

Theorem 10. *If $\mathfrak{B} = \{f_1, \ldots, f_n\}$ is a basis for a vector space V, then the* coordinate transformation $L_{\mathfrak{B}}(f) = [f]_{\mathfrak{B}}$ *is an isomorphism. In other words, every n-dimensional vector space is isomorphic to $\mathbb{R}^n$.*

Example 11. Show that $T(A) = S^{-1}AS$, where $S = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, is an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

Theorem 12. *Let V and W be vector spaces. For 2–4, we assume that V and W are finite dimensional.*

1. *A linear transformation T from V to W isn a isomorphism if and only if* $\ker(T) = \{0\}$ *and* $\operatorname{im}(T) = W$.
2. *If V is isomorphic to W, then* $\dim(V) = \dim(W)$.
3. *Suppose T is a linear transformation from V to W with* $\ker(T) = \{0\}$ *and* $\dim(V) = \dim(W)$. *Then T is an isomorphism.*
4. *Suppose T is a linear transformation from V to W with* $\operatorname{im}(T) = W$ *and* $\dim(V) = \dim(W)$. *Then T is an isomorphism.*